

ON THE NULLSTELLENSATZ FOR STEIN SPACES AND REAL C -ANALYTIC SETS

FRANCESCA ACQUISTAPACE, FABRIZIO BROGLIA AND JOSÉ F. FERNANDO

ABSTRACT. In this work we prove the real Nullstellensatz for the ring $\mathcal{O}(X)$ of analytic functions on a global analytic set $X \subset \mathbb{R}^n$ in terms of the *saturation* in $\mathcal{O}(X)$ of Lojasiewicz's radical. Namely, given an ideal $\mathfrak{a} \subset \mathcal{O}(X)$, the ideal $\mathcal{I}(\mathcal{Z}(\mathfrak{a}))$ of the zero-set $\mathcal{Z}(\mathfrak{a})$ of \mathfrak{a} coincides with the saturation $\widetilde{\sqrt[\mathbb{R}]{\mathfrak{a}}}$ of the Lojasiewicz radical $\sqrt[\mathbb{R}]{\mathfrak{a}}$ of \mathfrak{a} . In connection with this we revisit the classical results concerning (Hilbert's) Nullstellensatz in the framework of (complex) Stein spaces. We also prove that if the zero-set $\mathcal{Z}(\mathfrak{a})$ of \mathfrak{a} has “good properties” concerning Hilbert's 17th Problem, then $\mathcal{I}(\mathcal{Z}(\mathfrak{a})) = \widetilde{\sqrt[\mathbb{R}]{\mathfrak{a}}}$, where $\sqrt[\mathbb{R}]{\mathfrak{a}}$ stands for the *real radical* of \mathfrak{a} ; furthermore, the same holds if we replace $\sqrt[\mathbb{R}]{\mathfrak{a}}$ by the *analytic real radical* $\sqrt[\mathbb{A}]{\mathfrak{a}}$ of \mathfrak{a} , which is a natural generalization of the real radical ideal in the global analytic setting. We also study the relationship between a *normal primary decomposition* of a saturated ideal \mathfrak{a} of $\mathcal{O}(\mathbb{R}^n)$ and the decomposition as the union of its irreducible components of the germ $Y_{\mathbb{R}^n}$ of the support of the coherent sheaf which extends $\mathfrak{a}\mathcal{O}_{\mathbb{R}^n}$ to a suitable complex open neighborhood of \mathbb{R}^n . As one can expect, in case \mathfrak{a} is moreover prime, it holds that $\mathcal{I}(\mathcal{Z}(\mathfrak{a})) = \mathfrak{a}$ if and only if the (complex) dimension of the germ $Y_{\mathbb{R}^n}$ coincides with the (real) dimension of the global analytic set $\mathcal{Z}(\mathfrak{a})$.

INTRODUCTION

In this paper we characterize the ideals \mathfrak{a} of the algebra $\mathcal{O}(X)$ having the zero property, where X is either a Stein space or a C -analytic set (that is, a global analytic subset of \mathbb{R}^n); recall that an ideal \mathfrak{a} of $\mathcal{O}(X)$ has the *zero property* if it coincides with the ideal $\mathcal{I}(\mathcal{Z}(\mathfrak{a}))$ of all global analytic functions on X vanishing on its zero-set $\mathcal{Z}(\mathfrak{a})$. More generally, we approach the problem of determining algebraically the ideal $\mathcal{I}(\mathcal{Z}(\mathfrak{a}))$ from an ideal \mathfrak{a} of $\mathcal{O}(X)$; as it is well-known these problems are commonly known as Nullstellensätze. Both cases, the complex and the real analytic, have deserved for long the attention of specialists in both matters.

For the general real case our results are new and we recall that until the moment all known results were just for two particular situations: (1) compact analytic spaces (see [Jw, Rz]), or (2) analytic spaces of low dimensions: 0, 1 or 2 (see [A, BP]). On the other hand, for the complex case we extend the classical Forster's Nullstellensatz by removing the condition that the involved ideal \mathfrak{a} , for which is computed $\mathcal{I}(\mathcal{Z}(\mathfrak{a}))$, is closed. Let us be more precise.

Date: July 4th, 2012.

2000 Mathematics Subject Classification. Primary 32C05, 11E25, 26E05; Secondary 32C07, 32C15.

Key words and phrases. Nullstellensatz, Stein space, closed ideal, radical, real Nullstellensatz, real global analytic set, saturated ideal, Lojasiewicz's radical, convex ideal, H -sets, H^a -set, real ideal, real radical, real analytic ideal, analytic real radical, quasi-real ideal.

Authors supported by Spanish GAAR MTM2011-22435. First and second authors also supported by Italian GNSAGA of INdAM and MIUR.

The complex case. The main known results concerning the complex analytic Nullstellensatz go back to the sixties and are due to Forster [F] and to Siu [S] in the case of prime ideals. To state the main results we fix a Stein algebra $\mathcal{O}(X) = H^0(X, \mathcal{O}_X)$, that is, the algebra of global analytic sections on a (reduced) Stein space (X, \mathcal{O}_X) . There are crucial differences concerning the behaviour of polynomial functions on an algebraic variety and analytic functions on a Stein space. Besides that $\mathcal{O}(X)$ is neither noetherian nor a unique factorization domain, two main obstructions appear to get a Nullstellensatz. The first one arises because there are proper prime ideals with empty zero-set, while the second one appears because the “multiplicity” of an analytic function $G \in \mathcal{O}(X)$ vanishing (identically) on a discrete set can be unbounded; hence, if another analytic function $F \in \mathcal{O}(X)$ vanishes on the zero-set of G with multiplicity 1, no power of F can belong to the ideal $G\mathcal{O}(X)$. Classical examples of the previous situations, for which \mathbb{K} denotes either \mathbb{R} or \mathbb{C} , are the following:

Example 1. Let \mathfrak{A} be an ultrafilter of subsets of \mathbb{N} containing all cofinite subsets. For an analytic function $F \in \mathcal{O}(\mathbb{K})$, we denote by $\text{mult}_z(F)$ the *multiplicity* of F at the point $z \in \mathbb{K}$. Put $M(F, m) := \{n \in \mathbb{N} : \text{mult}_n(F) \geq m\}$. Consider the non-empty set

$$\mathfrak{a} := \{F \in \mathcal{O}(\mathbb{K}) : M(F, m) \in \mathfrak{A} \ \forall m \geq 0\}.$$

Let us check that \mathfrak{a} is a prime ideal. Indeed, if $F, G \in \mathfrak{a}$, then $M(F, m) \cap M(G, m) \subset M(F+G, m)$, because $\text{mult}_n(F+G) \geq \min\{\text{mult}_n(F), \text{mult}_n(G)\}$, and so $M(F+G, m) \in \mathfrak{A}$ for all $m \geq 0$. On the other hand, if $F \in \mathfrak{a}$ and $G \in \mathcal{O}(K)$ then $\text{mult}_n(FG) = \text{mult}_n(F) + \text{mult}_n(G)$ and so $M(FG, m) \supset M(F, m) \in \mathfrak{A}$ for all $m \geq 0$.

Next, suppose that $F_1 F_2 \in \mathfrak{a}$ but $F_1, F_2 \notin \mathfrak{a}$. Thus, there exists $m_1, m_2 \geq 0$ such that $M(F_1, m_1), M(F_2, m_2) \notin \mathfrak{A}$. Take $m_0 := \max\{m_1, m_2\}$ and note that the sets $M(F_1, m_0), M(F_2, m_0) \notin \mathfrak{A}$; hence, $\mathbb{N} \setminus (M(F_1, m_0) \cup M(F_2, m_0)) \in \mathfrak{A}$ but $M(F_1, m_0) \cup M(F_2, m_0) \subset M(F_1 F_2, m_0) \in \mathfrak{A}$, a contradiction. Thus, \mathfrak{a} is a prime ideal.

Finally, observe that $\mathcal{Z}(\mathfrak{a}) = \emptyset$. For each $k \geq 1$, let $G_k \in \mathcal{O}(\mathbb{K})$ be an analytic function such that $\mathcal{Z}(G_k) = \{n \in \mathbb{N} : n \geq k\}$ and $\text{mult}_n(G_k) = 2n$ for all $n \geq k$. Since \mathfrak{A} contains all cofinite subsets, we deduce that each $G_k \in \mathfrak{a}$ and so $\mathcal{Z}(\mathfrak{a}) \subset \bigcap_{k \geq 1} \mathcal{Z}(G_k) = \emptyset$.

Example 2. Let $F, G \in \mathcal{O}(\mathbb{K})$ be given by the infinite products:

$$F(z) := \prod_{n \geq 1} \left(1 - \frac{z}{n^2}\right) \quad \text{and} \quad G(z) := \prod_{n \geq 1} \left(1 - \frac{z}{n^2}\right)^n,$$

for all $z \in \mathbb{K}$. It holds that the zero-sets of F and G coincide with the set $\{n^2 : n \in \mathbb{N}^+\}$; we denote $\mathfrak{a} = G\mathcal{O}(\mathbb{K})$. Now, if the classical Nullstellensatz held for $\mathcal{O}(\mathbb{K})$ there would exist an integer $m \geq 0$ and a global analytic function $H \in \mathcal{O}(\mathbb{K})$ and, such that

$$F^m = GH.$$

Let us compare now multiplicities in the previous formula at the point $(m+1)^2$: the left hand side vanishes at the point $(m+1)^2$ with multiplicity m , while the right hand side vanishes at the point $(m+1)^2$ with multiplicity $m+1$, a contradiction. Thus, we conclude $\mathcal{I}(\mathcal{Z}(\mathfrak{a})) \neq \sqrt{\mathfrak{a}}$.

To control these difficulties, Forster showed firstly that the prime *closed* ideals \mathfrak{p} of $\mathcal{O}(X)$, endowed with its usual Frechet’s topology (see [GR, VIII.A]), have the zero property, that is, $\mathcal{I}(\mathcal{Z}(\mathfrak{p})) = \mathfrak{p}$, and afterwards that the closed ideals \mathfrak{a} of $\mathcal{O}(X)$ admit (as happens in

the noetherian case) a *normal primary decomposition* (see §1.3); of course, for a general normal primary decomposition there are countably many primary ideals \mathfrak{q}_i .

In this context we extend Forster's Nullstellensatz (see Section 2 for precise statements) to the non closed case as we state in the following result. In what follows, given an ideal \mathfrak{b} of $\mathcal{O}(X)$, we denote by $\overline{\mathfrak{b}}$ its closure with respect to the usual Frechet's topology of $\mathcal{O}(X)$.

Theorem 1 (Nullstellensatz). *Let (X, \mathcal{O}_X) be a Stein space and $\mathfrak{a} \subset \mathcal{O}(X)$ be an ideal. Then,*

$$\mathcal{I}(\mathcal{Z}(\mathfrak{a})) = \overline{\sqrt{\mathfrak{a}}}.$$

In particular,

- (i) $\mathcal{I}(\mathcal{Z}(\mathfrak{a})) = \sqrt{\mathfrak{a}}$ if and only if $\sqrt{\mathfrak{a}}$ is closed; and
- (ii) $\mathcal{I}(\mathcal{Z}(\mathfrak{a})) = \mathfrak{a}$ if and only if \mathfrak{a} is radical and closed.

Notice that for a closed ideal \mathfrak{a} with locally finite irredundant primary decomposition $\mathfrak{a} = \bigcap_{i \in I} \mathfrak{q}_i$, we have

$$\overline{\sqrt{\mathfrak{a}}} = \bigcap_{i \in I} \sqrt{\mathfrak{q}_i} \supset \sqrt{\mathfrak{a}}$$

because as we will see in Section 2 the radical of a closed ideal \mathfrak{a} needs not to be closed; however, the radical of a closed primary ideal \mathfrak{q} is still closed.

The real case. The situation in the real case is more delicate. First of all we have similar difficulties to the ones described in the complex analytic case; Examples 1 and 2 can be generalized to the real case.

Examples 3. (1) Notice first that the ideal \mathfrak{a} in Example 1 is a *real* ideal, that is, if a sum of squares $\sum_{i=1}^p f_i^2$ in $\mathcal{O}(\mathbb{R})$ belongs to \mathfrak{a} then each $f_i \in \mathfrak{a}$. Indeed, assume that $f := \sum_{i=1}^p f_i^2 \in \mathfrak{a}$. Since

$$\text{mult}_n(f) = 2 \min\{\text{mult}_n(f_1), \dots, \text{mult}_n(f_p)\},$$

we deduce that $M(f, 2m) \subset M(f_i, m)$ for all $m \geq 0$ and $i = 1, \dots, p$. Thus, since each $M(f, 2m) \in \mathfrak{A}$, we deduce that $M(f_i, m) \in \mathfrak{A}$ for all $m \geq 0$, that is, each $f_i \in \mathfrak{a}$. Thus, \mathfrak{a} is a real prime ideal with empty zero-set.

(2) Concerning Example 2, let us check that $\mathcal{I}(\mathcal{Z}(\mathfrak{a})) \neq \sqrt{\mathfrak{a}}$, where $\sqrt{\mathfrak{a}}$ stands for the real radical of \mathfrak{a} (see (I.2) below). To that end, let us check that the analytic function f does not belong to $\sqrt{\mathfrak{a}} = \sqrt{g\mathcal{O}(\mathbb{R})}$, where $f, g \in \mathcal{O}(\mathbb{R})$ are given by the corresponding formulae proposed in Example 2. Otherwise, there would exist an integer $m \geq 1$ and analytic functions $h_1, \dots, h_p, h \in \mathcal{O}(\mathbb{R})$ such that

$$f^{2m} + \sum_{i=1}^p h_i^2 = gh.$$

Comparing orders in both sides of the previous equality at the point $(2m+1)^2$, we achieve a contradiction.

Consider now a global analytic subset $X \subset \mathbb{R}^n$ and let $\mathcal{I}(X)$ be the ideal of all global (real) analytic functions vanishing on X . The structural sheaf of X is the coherent sheaf $\mathcal{O}_X := \mathcal{O}_{\mathbb{R}^n}/\mathcal{I}(X)\mathcal{O}_{\mathbb{R}^n}$ and its ring of global real analytic sections $\mathcal{O}(X) := H^0(X, \mathcal{O}_X) = \mathcal{O}(\mathbb{R}^n)/\mathcal{I}(X)$ can be seen as a subset of the Stein algebra $\mathcal{O}(\tilde{X})$ of its *complexification* \tilde{X} (understood as a complex analytic set germ at X , see §1.1). Note that we are not saying

that X is coherent as an analytic set. Recall also that Cartan proved in [C1, VIII.Thm.4, pag.60] that if Y is a Stein space, the closure of an ideal \mathfrak{b} of $\mathcal{O}(Y)$ coincides with its *saturation* (via the sheaf \mathcal{O}_Y):

$$\tilde{\mathfrak{b}} := \{F \in \mathcal{O}(Y) : F_z \in \mathfrak{b}\mathcal{O}_{Y,z} \ \forall z \in Y\}.$$

Thus, $\mathcal{O}(X)$ inherits the induced topology of $\mathcal{O}(\tilde{X})$, but now the saturation (via the sheaf \mathcal{O}_X):

$$\tilde{\mathfrak{a}} := \{f \in \mathcal{O}(X) : f_x \in \mathfrak{a}\mathcal{O}_{X,x} \ \forall x \in X\}.$$

of an ideal \mathfrak{a} of $\mathcal{O}(X)$ needs not to be closed. Nevertheless, as de Bartolomeis proved in [dB1, dB2], each saturated ideal of $\mathcal{O}(X)$ admits a *normal primary decomposition* similar to the one in the complex case. Note also that the previous definition of saturation coincides with the one of Whitney for ideals in the ring of smooth functions over a real smooth manifold (see [M, II.1.3]).

Before stating our main result, we introduce some terminology. All in what follows, given $f, g \in \mathcal{O}(X)$ we say that $f \geq g$ if $f(x) \geq g(x)$ for all $x \in X$. Now, given an ideal \mathfrak{a} of $\mathcal{O}(X)$, we define its *Łojasiewicz radical* as:

$$\sqrt[p]{\mathfrak{a}} := \{g \in \mathcal{O}(X) : \exists f \in \mathfrak{a} \ \& \ m \geq 1 \text{ such that } f - g^{2m} \geq 0\}. \quad (\text{I.1})$$

The notion of Łojasiewicz radical has been used by many authors to approach different problems mainly related to rings of germs of different types, see for instance [N], [D, pag. 104], [K, 1.21] or [DM, §6] between others. More generally, we say that an ideal \mathfrak{a} of $\mathcal{O}(X)$ is *convex* if each $g \in \mathcal{O}(X)$ satisfying $|g| \leq f$ for some $f \in \mathfrak{a}$ belongs to \mathfrak{a} . In particular, the Łojasiewicz's radical $\sqrt[p]{\mathfrak{a}}$ of an ideal \mathfrak{a} of $\mathcal{O}(X)$ is a radical convex ideal. Our main theorem is the following.

Theorem 2 (Real Nullstellensatz). *Let $X \subset \mathbb{R}^n$ be a global analytic set and let \mathfrak{a} be an ideal of the ring $\mathcal{O}(X)$. Then,*

$$\mathcal{I}(\mathcal{Z}(\mathfrak{a})) = \widetilde{\sqrt[p]{\mathfrak{a}}}.$$

In particular,

- (i) $\mathcal{I}(\mathcal{Z}(\mathfrak{a})) = \sqrt[p]{\mathfrak{a}}$ if and only if $\sqrt[p]{\mathfrak{a}}$ is a saturated ideal,
- (ii) $\mathcal{I}(\mathcal{Z}(\mathfrak{a})) = \mathfrak{a}$ if and only if \mathfrak{a} is a convex, radical and saturated ideal.

If we compare the previous result with the real Nullstellensatz for the ring of polynomial functions on a real algebraic variety, we observe that the Łojasiewicz radical plays the same role of the classical *real radical*, defined in our context for an ideal \mathfrak{a} of $\mathcal{O}(X)$ as:

$$\sqrt{\mathfrak{a}} := \{f \in \mathcal{O}(X) : f^{2m} + \sum_{k=1}^p a_k^2 \in \mathfrak{a} \ \& \ a_i \in \mathcal{O}(X), \ m, p \geq 0\} \quad (\text{I.2})$$

It is natural to search the relations between both radicals and of course this question forces to confront positive semidefinite analytic functions versus sums of squares of analytic functions. It is important to note that in the abstract setting of the real spectrum of a ring A , both radicals coincide, see Section 3. In Section 5 we prove the equality $\sqrt[p]{\mathfrak{a}} = \sqrt{\mathfrak{a}}$ for an ideal \mathfrak{a} of $\mathcal{O}(X)$ with the property that every positive semidefinite analytic function whose zero-set is $Z := \mathcal{Z}(\mathfrak{a})$ can be represented as a (finite) sum of squares of meromorphic functions on X ; for short, any global analytic set $Z \subset \mathbb{R}^n$ with the previous property will be called *H-set*. Some examples of *H-sets* are the following: discrete sets (see [BKS]) and

compact sets (see [Jw, Rz]). Moreover, if X is either an analytic curves or a coherent analytic surface, every analytic subset of X is an H -set (see [ABFR1, ABFR2]).

Also, since infinite (convergent) sums of squares of meromorphic functions make sense in $\mathcal{O}(X)$ (see Section 1 and [ABF, ABFR3]), we define the *analytic real radical* of an ideal \mathfrak{a} of $\mathcal{O}(X)$ (see also [BP]) as

$$\sqrt[\mathfrak{r}]{\mathfrak{a}} := \left\{ f \in \mathcal{O}(\mathbb{R}^n) : f^{2m} + \sum_{k \geq 1} a_k^2 \in \mathfrak{a} \text{ \& } a_i \in \mathcal{O}(\mathbb{R}^n), m \geq 0 \right\} \quad (\text{I.3})$$

As before the equality $\sqrt[\mathfrak{r}]{\mathfrak{a}} = \sqrt[\mathfrak{l}]{\mathfrak{a}}$ holds for an ideal \mathfrak{a} of $\mathcal{O}(X)$ with the property that every positive semidefinite analytic function whose zero-set is $Z := \mathcal{Z}(\mathfrak{a})$ can be represented as a infinite sum of squares of meromorphic functions on X ; analogously, we call $H^{\mathfrak{a}}$ -sets those global analytic sets with the previous property. An example of $H^{\mathfrak{a}}$ -set is the countable union of disjoint compact analytic sets. Observe that if all the connected components of X are compact, then all the global analytic subsets of X are $H^{\mathfrak{a}}$ -sets.

Thus, for the previous situation we get the following result, proved in Section 5.

Theorem 3. *Let $X \subset \mathbb{R}^n$ be a global analytic set and let \mathfrak{a} be an ideal of $\mathcal{O}(X)$ such that $\mathcal{Z}(\mathfrak{a})$ is an H -set. Then,*

$$\mathcal{I}(\mathcal{Z}(\mathfrak{a})) = \widetilde{\sqrt[\mathfrak{r}]{\mathfrak{a}}}.$$

In particular,

- (i) $\mathcal{I}(\mathcal{Z}(\mathfrak{a})) = \sqrt[\mathfrak{r}]{\mathfrak{a}}$ if and only if $\sqrt[\mathfrak{r}]{\mathfrak{a}}$ is saturated.
- (ii) $\mathcal{I}(\mathcal{Z}(\mathfrak{a})) = \mathfrak{a}$ if and only if \mathfrak{a} is real and saturated.
- (iii) If \mathfrak{a} is prime then $\mathcal{I}(\mathcal{Z}(\mathfrak{a})) = \mathfrak{a}$ if and only if \mathfrak{a} is a real saturated prime ideal.

Moreover, the statements above hold if $\mathcal{Z}(\mathfrak{a})$ is an $H^{\mathfrak{a}}$ -set by replacing the real radical $\sqrt[\mathfrak{r}]{\mathfrak{a}}$ by the analytic real radical $\widetilde{\sqrt[\mathfrak{r}]{\mathfrak{a}}}$.

In particular, the previous result applies if X is either an analytic curve, a coherent analytic surface, or a global analytic set whose connected components are all compact, and so the real Nullstellensatz holds for such an X in terms of the real radical (or more generally, of the analytic real radical).

Finally, in Section 6 we prove that the class of ideals of $\mathcal{O}(X)$ which have the zero property enjoys the good expected properties, as it happens with the corresponding class in the algebraic setting. More precisely, we prove:

Theorem 4. *Let $\mathfrak{q} \subset \mathcal{O}(\mathbb{R}^n)$ be a saturated primary ideal. Then, the following assertions are equivalent*

- (i) $\mathcal{I}(\mathcal{Z}(\mathfrak{q})) = \sqrt[\mathfrak{r}]{\mathfrak{q}}$.
- (ii) $\dim \mathcal{Z}_{\mathbb{C}}(\mathfrak{q}) = \dim \mathcal{Z}(\mathfrak{q})$.
- (iii) There exists $x \in \mathcal{Z}(\mathfrak{q})$ such that $\dim \mathcal{Z}(\mathfrak{q}\mathcal{O}_{\mathbb{R}^n, x}) = \dim \mathcal{Z}(\mathfrak{q}\mathcal{O}_{\mathbb{C}^n, x})$.

As it is well-known condition (iii) in Theorem 4 is equivalent to the existence of a regular point $y \in \mathcal{Z}(\mathfrak{q})$ for the ideal $\sqrt[\mathfrak{r}]{\mathfrak{q}}$; recall that $y \in \mathcal{Z}(\mathfrak{q})$ is *regular* for the ideal $\sqrt[\mathfrak{r}]{\mathfrak{q}}$ if $\dim \mathcal{Z}(\mathfrak{q})_y = k$ and there exists $f_{k+1}, \dots, f_n \in \sqrt[\mathfrak{r}]{\mathfrak{q}}$ such that $\text{rk}(\nabla f_{k+1}(y), \dots, \nabla f_n(y)) = n - k$; notice that the two previous conditions imply in particular that $\mathcal{Z}(\mathfrak{q}) \cap U = \mathcal{Z}(f_{k+1}, \dots, f_n) \cap U$ in a neighborhood U of x .

The paper is organized as follows. In Section 1 we recall Forster's and de Bartolomeis' normal primary decompositions for saturated ideals and we recall what we mean precisely

by infinite sums of squares in the analytic setting. Section 2 is devoted to the complex Nullstellensatz, while the real Nullstellensatz is the argument of Section 4. In Section 3 we see that in the abstract setting the Łojasiewicz radical and the real radical coincide. In Section 5 we prove that an affirmative answer for Hilbert's 17th Problem for the global analytic case implies that Łojasiewicz's radical and the real radical coincide in this setting. We also discuss certain properties concerning *convex* and *quasi-real* ideals. Finally, in Section 6 we analyze the geometric meaning of having a real Nullstellensatz for the ideal \mathfrak{a} , comparing the real dimension of the global analytic set $Z := \mathcal{Z}(\mathfrak{a})$ and the complex dimension of the germ $\mathcal{Z}(\mathfrak{a} \otimes \mathbb{C})$.

1. PRELIMINARIES ON ANALYTIC GEOMETRY AND SATURATED IDEALS

Although we mainly deal with real analytic functions, we will of course use complex analysis. All through this work holomorphic functions will refer to the complex case and analytic functions to the real case. For further readings about holomorphic functions we refer the reader to the classical [GR]. We begin by introducing some notations.

1.1. General terminology. Denote the coordinates in \mathbb{C}^n by $z := (z_1, \dots, z_n)$, with $z_i := x_i + \sqrt{-1}y_i$, where $x_i := \operatorname{Re}(z_i)$ and $y_i := \operatorname{Im}(z_i)$ are respectively the *real* and the *imaginary parts* of z_i . Consider the usual conjugation $\sigma : \mathbb{C}^n \rightarrow \mathbb{C}^n$, $z \mapsto \bar{z} := (\bar{z}_1, \dots, \bar{z}_n)$, whose set of fixed points is \mathbb{R}^n . A subset $A \subset \mathbb{C}^n$ is *invariant* if $\sigma(A) = A$; obviously, $A \cap \sigma(A)$ is the biggest invariant subset of A . Let $\Omega \subset \mathbb{C}^n$ be an invariant open set and let $F : \Omega \rightarrow \mathbb{C}$ be a holomorphic function. We say that F is *invariant* if $F(z) = \overline{(F \circ \sigma)(z)}$ for all $z \in \Omega$. This implies that F restricts to a (real) analytic function on $\Omega \cap \mathbb{R}^n$. Conversely, if f is analytic on \mathbb{R}^n , it extends to an invariant holomorphic function F on some invariant open neighbourhood Ω of \mathbb{R}^n . In general, we denote by:

$$\Re(F) : \Omega \rightarrow \mathbb{C}, \quad z \mapsto \frac{F(z) + \overline{(F \circ \sigma)(z)}}{2} \quad \text{and} \quad \Im(F) : \Omega \rightarrow \mathbb{C}, \quad z \mapsto \frac{F(z) - \overline{(F \circ \sigma)(z)}}{2\sqrt{-1}}$$

the *real* and the *imaginary parts* of F , which satisfy $F = \Re(F) + \sqrt{-1}\Im(F)$. Recall also that an analytic subsheaf \mathcal{F} of \mathcal{O}_Ω is called *invariant* if for each $F \in \mathcal{F}$, the holomorphic function $\overline{F \circ \sigma} \in \mathcal{F}$. Observe that if \mathcal{F} is a invariant sheaf on Ω and $F_1, \dots, F_r \in H^0(\Omega, \mathcal{F})$ generate \mathcal{F}_z as a $\mathcal{O}_{\Omega, z}$ -module for some $z \in \Omega$, then also $\Re(F_1), \Im(F_1), \dots, \Re(F_r), \Im(F_r)$ generate \mathcal{F}_z as a $\mathcal{O}_{\Omega, z}$ -module.

Along this work we will use the symbol $\mathcal{Z}(\cdot)$ to denote the zero-set of (\cdot) and $\mathcal{I}(\cdot)$ to denote the ideal of functions vanishing identically on (\cdot) . For instance, if (X, \mathcal{O}_X) is either a Stein space or a real coherent analytic space and $S \subset \mathcal{O}(X)$, the *zero-set* of S is $\mathcal{Z}(S) := \{x \in X : F(x) = 0 \ \forall F \in S\}$, while if $Z \subset X$ the *ideal* of Z is $\mathcal{I}(Z) := \{F \in \mathcal{O}(X) : F(x) = 0 \ \forall x \in Z\}$. In what follow, for the sake of clearness we denote the elements of $\mathcal{O}(X)$ by using capital letters if (X, \mathcal{O}_X) is a Stein space and by using small letters if (X, \mathcal{O}_X) is a real coherent analytic space. However, if a property holds for both type of space we will keep capital letters.

Recall at this point also that if (X, \mathcal{O}_X) is a coherent (paracompact) real analytic space, there exists a (paracompact) complex analytic space $(\tilde{X}, \mathcal{O}_{\tilde{X}})$ such that:

- (i) $X \subset \tilde{X}$ is a closed subset and $\mathcal{O}_{\tilde{X}, x} = \mathcal{O}_{X, x} \otimes \mathbb{C}$ for all $x \in X$.
- (ii) There exists an antiholomorphic involution $\sigma : \tilde{X} \rightarrow \tilde{X}$ whose fixed locus is X .
- (iii) X has in \tilde{X} a fundamental system of invariant open Stein neighbourhoods.

(iv) If X is reduced so is \tilde{X} .

The analytic space $(\tilde{X}, \mathcal{O}_{\tilde{X}})$ is called a *complexification* of X . It holds also that the germ of $(\tilde{X}, \mathcal{O}_{\tilde{X}})$ at X is unique up to isomorphism. For further details concerning the complexification of a real analytic space, see [C2, To, WB].

Note also that a global analytic set $X \subset \mathbb{R}^n$ endowed with its (coherent) structural sheaf $\mathcal{O}_X = \mathcal{O}_{\mathbb{R}^n}/\mathcal{I}(X)\mathcal{O}_{\mathbb{R}^n}$ has a well defined complexification exactly as above, except perhaps for the second condition in (i). Note also that in this case (X, \mathcal{O}_X) is automatically reduced. From now on by real analytic space we will mean a global analytic set in \mathbb{R}^n endowed with its structural sheaf.

1.2. Saturated and closed ideals. Let (X, \mathcal{O}_X) be either a Stein space or a real analytic space and $\mathfrak{a} \subset \mathcal{O}(X)$ be an ideal. We consider its *saturation*

$$\tilde{\mathfrak{a}} := \{F \in \mathcal{O}(X) : F_x \in \mathfrak{a}\mathcal{O}_{X,x} \ \forall x \in X\}.$$

Of course, the ideal \mathfrak{a} is *saturated* if $\tilde{\mathfrak{a}} = \mathfrak{a}$.

As Cartan proved in [C1, VIII.Thm.4, pag.60], in the complex case $\tilde{\mathfrak{a}}$ coincides with the closure of \mathfrak{a} in $\mathcal{O}(X)$ endowed with its usual Frechet topology; hence, saturated ideals coincide with closed ideals. Recall that in case (X, \mathcal{O}_X) is a reduced Stein space, its Frechet topology is induced by a countable collection of the natural seminorms $\|\cdot\|_m := \sup_{K_m} \{|\cdot|\}$, where $\{K_m\}_{m \geq 1}$ is an exhaustion of X by compact sets; of course, this topology does not depend on the chosen exhaustion, see [GR, VIII.A].

On the other hand, if (X, \mathcal{O}_X) is a real analytic space the suitable topology is the one induced by the following convergence: *A sequence $\{f_k\}_{k \geq 1}$ of elements of $\mathcal{O}(X)$ converge to $f \in \mathcal{O}(X)$ if there exist a complexification $(\tilde{X}, \mathcal{O}_{\tilde{X}})$ of (X, \mathcal{O}_X) and holomorphic extensions F_k of f_k and F of f such that F_k converges to F in $H^0(\tilde{X}, \mathcal{O}_{\tilde{X}})$ endowed with its Frechet topology* (see [dB1, §1.5]). With this topology $\mathcal{O}(X)$ is a complete topological \mathbb{R} -algebra.

The saturation arises “naturally” when dealing with Nullstellensätze to manage the existence of proper prime ideals and proper real prime ideals with empty zero-set (see Examples 1 and 3 in the Introduction).

1.3. Closed primary ideals and normal primary decomposition. Let (X, \mathcal{O}_X) be either a Stein space or a real analytic space. One of the main properties of the closed and saturated ideals of $\mathcal{O}(X)$ is that they enjoy a locally finite primary decomposition. Before entering into further details we recall some preliminary definitions. Given a collection of ideals $\mathcal{A} := \{\mathfrak{a}_i\}_{i \in I}$ of $\mathcal{O}(X)$, we say that \mathcal{A} is *locally finite*, if the family of their zero-sets $\{\mathcal{Z}(\mathfrak{a}_i)\}_{i \in I}$ is locally finite in X . A decomposition $\mathfrak{a} = \bigcap_{i \in I} \mathfrak{a}_i$ of an ideal \mathfrak{a} of $\mathcal{O}(X)$ is called *irredundant* if $\mathfrak{a} \neq \bigcap_{i \in K} \mathfrak{a}_i$ for each proper subset $K \subsetneq I$. Moreover, a primary decomposition $\mathfrak{a} = \bigcap_{i \in I} \mathfrak{q}_i$ of an ideal \mathfrak{a} of $\mathcal{O}(X)$ is called *normal* if it is locally finite, irredundant and the associated prime ideals $\mathfrak{p}_i := \sqrt{\mathfrak{q}_i}$ are pairwise distinct. As usual, a primary ideal $\mathfrak{q}_j \in \{\mathfrak{q}_i\}_{i \in I}$ is called an *isolated primary component* if \mathfrak{p}_j is minimal among the primes $\{\mathfrak{p}_i\}_{i \in I}$; otherwise, \mathfrak{q}_j is an *immersed primary component*.

Before we present the normal primary decomposition of saturated ideals due to Forster, we recall here some results concerning saturated primary and prime ideals which provides a clear idea of their behaviour.

Lemma 1.1. *Let $\mathfrak{q} \subset \mathcal{O}(X)$ be a primary ideal and let $F \in \mathcal{O}(X)$. We have:*

- (i) If $x \in \mathcal{Z}(\mathfrak{q})$, then $F \in \mathfrak{q}$ if and only if $F_x \in \mathfrak{q}\mathcal{O}_{X,x}$.
- (ii) \mathfrak{q} is saturated if and only if $\mathcal{Z}(\mathfrak{q}) \neq \emptyset$.
- (iii) $\mathcal{Z}(\mathfrak{q})$ is connected.

Proof. (i) See [F, §3.1.Lem.] and [dB1, 2.1.2]. In the statement of both results the authors assume that the ideal \mathfrak{q} is saturated, but this fact is only used to assure that $\mathcal{Z}(\mathfrak{q}) \neq \emptyset$.

(ii) The “only if” implication is clear. For the converse, choose a point $x \in \mathcal{Z}(\mathfrak{q})$ and observe that by (i), $\mathfrak{q} = \{F \in \mathcal{O}(X) : F_x \in \mathfrak{q}\mathcal{O}_{X,x}\}$; hence, \mathfrak{q} is the “saturation” of a local ideal and so it is saturated.

(iii) If (X, \mathcal{O}_X) is a Stein space, the result follows from Theorem 2.1. If (X, \mathcal{O}_X) is a real analytic space, we recall a classical trick. Assume, by way of contradiction, that $\mathcal{Z}(\mathfrak{q})$ is not connected and let $Y_1, Y_2 \subset \mathcal{Z}(\mathfrak{q})$ be two closed disjoint subsets such that $\mathcal{Z}(\mathfrak{q}) = Y_1 \cup Y_2$. Observe in particular that \mathfrak{q} must be saturated. Let $f \in \mathfrak{q}$ be such that $\mathcal{Z}(\mathfrak{q}) = \mathcal{Z}(f)$ (see Lemma 4.1 below) and let $g \in X$ be an analytic function such that g is strictly positive on Y_1 and strictly negative on Y_2 (to construct g use Whitney’s approximation lemma). Observe that $\mathcal{Z}(f^2 + g^2) = \emptyset$ and so $h_i = \sqrt{f^2 + g^2} + (-1)^i g$ is an analytic function whose zero-set is Y_i ; moreover, $h_1 h_2 = f^2 \in \mathfrak{q}$. However, $h_1, h_2 \notin \sqrt{\mathfrak{q}}$ because neither of them vanishes on $\mathcal{Z}(\mathfrak{q})$, a contradiction. Hence, $\mathcal{Z}(\mathfrak{q})$ is connected. \square

Lemma 1.2. ([F, §4.Hilfssatz 5] and [dB2, 2.2.10]) *Let $\{\mathfrak{a}_i\}_{i \in I} \subset \mathcal{O}(X)$ be a locally finite family of saturated ideals and let $\mathfrak{p} \subset \mathcal{O}(X)$ be a prime saturated ideal such that $\bigcap_{i \in I} \mathfrak{a}_i \subset \mathfrak{p}$. Then, there exists $i \in I$ such that $\mathfrak{a}_i \subset \mathfrak{p}$.*

Now, we recall the normal primary decomposition of saturated ideals of $\mathcal{O}(X)$; see [F, §5] and [dB1, Thm. 2.3.6] for further details.

Proposition 1.3. *Let $\mathfrak{a} \subset \mathcal{O}(X)$ be a saturated ideal of $\mathcal{O}(X)$. Then, \mathfrak{a} admits a normal primary decomposition $\mathfrak{a} = \bigcap_i \mathfrak{q}_i$ such that all the primary ideals \mathfrak{q}_i are saturated. Moreover, the prime ideals $\mathfrak{p}_i := \sqrt{\mathfrak{q}_i}$ and the primary isolated components are uniquely determined by \mathfrak{a} and do not depend on the normal primary decomposition of \mathfrak{a} .*

As the reader can straightforwardly check the normal primary decompositions enjoy the good behaviour one can expect for radical, real and real analytic ideals. Namely,

Corollary 1.4. *Let $\mathfrak{a} \subset \mathcal{O}(X)$ be a saturated ideal and let $\mathfrak{a} = \bigcap_i \mathfrak{q}_i$ be a normal primary decomposition of \mathfrak{a} . We have:*

- (i) *If \mathfrak{a} is a radical, each \mathfrak{q}_i is prime. Moreover, if such is the case the normal primary decomposition is unique.*
- (ii) *If \mathfrak{a} is a real analytic (resp. real) ideal, every \mathfrak{q}_i is a real analytic (resp. real) prime ideal. Again, under these hypotheses the normal primary decomposition is unique.*

1.4. Infinite sum of squares. Let (X, \mathcal{O}_X) be a real analytic space. Following what is proposed in [ABFR3, 1.3] for a real analytic manifold, we say that an element $f \in \mathcal{O}(X)$ is an *infinite sum of squares of meromorphic functions on X* if there exist a non zero divisor $g \in \mathcal{O}(X)$ such that $g^2 f$ is an absolutely convergent series $\sum_{k \geq 1} f_k^2$ in $\mathcal{O}(X)$, that is, there exist a complexification $(\tilde{X}, \mathcal{O}_{\tilde{X}})$ of (X, \mathcal{O}_X) and holomorphic extensions F_k of f_k , F of f and G of g such that $G^2 F = \sum_{k \geq 1} F_k^2$ and $\sum_{k \geq 1} F_k^2$ is a absolutely convergent series with respect to the Frechet topology of $H^0(\tilde{X}, \mathcal{O}_{\tilde{X}})$; in other words, for each compact set

$K \subset \tilde{X}$ the series $\sum_{k \geq 1} \sup_K |F_k^2|$ is convergent. For further details concerning infinite sums of squares and Hilbert's 17th Problem for $\mathcal{O}(\mathbb{R}^n)$, see [ABF, ABFR3, Fe].

2. THE COMPLEX ANALYTIC HILBERT'S NULLSTELLENSATZ

The purpose of this section is to prove Theorem 1. First, we recall Forster's results concerning the Nullstellensatz for Stein algebras. We begin with the closed primary case.

Theorem 2.1 (Closed primary case). *Let (X, \mathcal{O}_X) be a Stein space and let $\mathfrak{q} \subset \mathcal{O}(X)$ be a closed primary ideal. Then,*

$$\mathcal{I}(\mathcal{Z}(\mathfrak{q})) = \sqrt{\mathfrak{q}}$$

Moreover,

- (i) *There exists a positive integer $m \geq 1$ such that $(\sqrt{\mathfrak{q}})^m \subset \mathfrak{q}$.*
- (ii) *In particular, if $\mathfrak{p} \subset \mathcal{O}(X)$ is a closed prime ideal, then $\mathcal{I}(\mathcal{Z}(\mathfrak{p})) = \mathfrak{p}$.*

Theorem 2.2 (Closed general case). *Let (X, \mathcal{O}_X) be a Stein space and let $\mathfrak{a} \subset \mathcal{O}(X)$ be a closed ideal. Consider a normal primary decomposition $\mathfrak{a} = \bigcap_{i \in I} \mathfrak{q}_i$ of \mathfrak{a} . For each $i \in I$, define*

$$\begin{aligned} \mathbf{h}(\mathfrak{q}_i, \mathfrak{a}) &:= \inf \left\{ k \in \mathbb{N} : F^k \in \mathfrak{q}_i, \forall F \in \overline{\sqrt{\mathfrak{a}}} \right\}, \\ \mathbf{h}(\mathfrak{q}_i) &:= \inf \{ k \in \mathbb{N} : F^k \in \mathfrak{q}_i, \forall F \in \sqrt{\mathfrak{q}_i} \}, \\ \mathbf{h}(\mathfrak{a}) &:= \inf \left\{ k \in \mathbb{N} : F^k \in \mathfrak{a}, \forall F \in \overline{\sqrt{\mathfrak{a}}} \right\}. \end{aligned}$$

Then, we have

- (i) $\mathbf{h}(\mathfrak{a}) = \sup_{i \in I} \{\mathbf{h}(\mathfrak{q}_i, \mathfrak{a})\}$
- (ii) $\sqrt{\mathfrak{a}}$ is closed if and only if $\mathbf{h}(\mathfrak{a}) < +\infty$.
- (iii) If \mathfrak{a} does not have immersed components, $\mathbf{h}(\mathfrak{a}) = \sup_{i \in I} \{\mathbf{h}(\mathfrak{q}_i)\}$.
- (iv) $\mathcal{I}(\mathcal{Z}(\mathfrak{a})) = \sqrt{\mathfrak{a}}$ if and only if $\mathbf{h}(\mathfrak{a}) < +\infty$, and if such is the case $\sqrt{\mathfrak{a}}^{\mathbf{h}(\mathfrak{a})} \subset \mathfrak{a}$.

To extend the above result to the non closed case we need the following characterization of the saturation of an ideal. Namely,

Definition and Lemma 2.3. *Let (X, \mathcal{O}_X) be either a Stein space or a real analytic space and let \mathfrak{a} be an ideal of $\mathcal{O}(X)$. Define*

$$\begin{aligned} \mathfrak{C}_1(\mathfrak{a}) &:= \{G \in \mathcal{O}(X) : \forall K \subset X \text{ compact } \exists H \in \mathcal{O}(X) \\ &\quad \text{such that } \mathcal{Z}(H) \cap K = \emptyset \text{ \& } HG \in \mathfrak{a}\}, \\ \mathfrak{C}_2(\mathfrak{a}) &:= \{G \in \mathcal{O}(X) : \forall x \in X \exists H \in H^0(X, \mathcal{O}_X) \\ &\quad \text{such that } H(x) \neq 0 \text{ \& } HG \in \mathfrak{a}\}. \end{aligned}$$

Then, $\tilde{\mathfrak{a}} = \mathfrak{C}_1(\mathfrak{a}) = \mathfrak{C}_2(\mathfrak{a})$.

Proof. The chain of inclusions $\mathfrak{C}_1(\mathfrak{a}) \subset \mathfrak{C}_2(\mathfrak{a}) \subset \tilde{\mathfrak{a}}$ is clear; hence, it only remains to check that $\tilde{\mathfrak{a}} \subset \mathfrak{C}_1(\mathfrak{a})$.

We begin with the complex case. Let $K \subset X$ be a compact set. Since (X, \mathcal{O}_X) is a Stein space, we may assume that K is holomorphically convex (see [GR, VII.A]). Since $\mathfrak{a}\mathcal{O}_X$ is a coherent sheaf we deduce, by Cartan's Theorem A (see [C2]), that there exists an open neighbourhood Ω of K in X and $A_1, \dots, A_r \in \mathcal{O}(X)$ such that $\mathfrak{a}\mathcal{O}_{X,x}$ is generated as an

$\mathcal{O}_{X,x}$ -module by $A_{1,x}, \dots, A_{r,x}$ for all $x \in \Omega$. As a consequence of Theorem B, we deduce, by [F, §2.Satz 3], that the finitely generated ideal $\mathfrak{g} := (A_1, \dots, A_r)\mathcal{O}(X)$ is saturated.

Moreover, by [F, §2.Satz 3], also the ideal

$$(\mathfrak{g} : \tilde{\mathfrak{a}}) := \{H \in \mathcal{O}(X) : H\tilde{\mathfrak{a}} \subset \mathfrak{g}\}$$

is saturated. Since $\mathfrak{a}\mathcal{O}_{X,x} = \tilde{\mathfrak{a}}\mathcal{O}_{X,x}$ for all $x \in X$, we deduce that $(\mathfrak{g} : \tilde{\mathfrak{a}})\mathcal{O}_{X,x} = \mathcal{O}_{X,x}$ for all $x \in \Omega$, that is, it is generated by 1 at any point of Ω . After shrinking Ω , we may assume that it is an Oka–Weil neighbourhood of K and that $H^0(W, (\mathfrak{g} : \tilde{\mathfrak{a}})\mathcal{O}_X) = H^0(W, \mathcal{O}_X)$ (see [GR, VII.A.Prop.3 & VIII.A.Prop.6]). Now, by [GR, VIII.A.Thm.11], there exist a holomorphic function $H \in H^0(X, (\mathfrak{g} : \tilde{\mathfrak{a}})\mathcal{O}_X) = (\mathfrak{g} : \tilde{\mathfrak{a}})$ which is close to 1 in K . Thus, $\mathcal{Z}(H) \cap K = \emptyset$ and $H\tilde{\mathfrak{a}} \subset \mathfrak{g} \subset \mathfrak{a}$. Therefore, we conclude that $\tilde{\mathfrak{a}} \subset \mathfrak{C}_1(\mathfrak{a})$.

Next we consider the real case. By [C2, Prop.2 & 5] the sheaf of ideals $\mathfrak{a}\mathcal{O}_X$ extends to a coherent sheaf of ideals \mathcal{F} on an open Stein neighborhood Ω of \mathbb{R}^n in \mathbb{C}^n . Hence the inclusion $\tilde{\mathfrak{a}} \subset \mathfrak{C}_1(\mathfrak{a})$ follows similarly to the one of the complex case and we leave the concrete details to the reader. \square

Remarks 2.4. Let $\mathfrak{a} \subset \mathfrak{b}$ be ideals of $\mathcal{O}(X)$ and define $\mathfrak{R}_i(\mathfrak{a}) := \mathfrak{C}_i(\sqrt{\mathfrak{a}})$ for $i = 1, 2$. Then,

- (1) $\mathfrak{C}_i(\mathfrak{a}) \subset \mathfrak{C}_i(\mathfrak{b})$ and $\mathfrak{R}_i(\mathfrak{a}) \subset \mathfrak{R}_i(\mathfrak{b})$.
- (2) $\mathfrak{C}_i(\mathfrak{C}_i(\mathfrak{a})) = \mathfrak{C}_i(\mathfrak{a})$ and $\mathfrak{R}_i(\mathfrak{R}_i(\mathfrak{a})) = \mathfrak{R}_i(\mathfrak{a})$.

Now, we are ready to prove Theorem 1.

Proof of Theorem 1. Let us prove that

$$\mathcal{I}(\mathcal{Z}(\mathfrak{a})) = \mathfrak{R}_1(\mathfrak{a}) := \mathfrak{C}_1(\sqrt{\mathfrak{a}}) = \mathfrak{R}_2(\mathfrak{a}) := \mathfrak{C}_2(\sqrt{\mathfrak{a}}) = \widetilde{\sqrt{\mathfrak{a}}}.$$

Clearly, $\mathfrak{R}_1(\mathfrak{a}) \subset \mathfrak{R}_2(\mathfrak{a}) \subset \widetilde{\sqrt{\mathfrak{a}}} \subset \mathcal{I}(\mathcal{Z}(\mathfrak{a}))$. Thus, it only remains to prove the inclusion

$$\mathcal{I}(\mathcal{Z}(\mathfrak{a})) \subset \mathfrak{R}_1(\mathfrak{a}).$$

Assume first that \mathfrak{a} is a closed ideal and let K be a compact subset of X . Since (X, \mathcal{O}_X) is a Stein space, we may assume that K is holomorphically convex (see [GR, VII.A]). Let $\mathfrak{a} = \bigcap_{i \in I} \mathfrak{q}_i$ be a normal primary decomposition of \mathfrak{a} . Since K is compact and $\{\mathfrak{q}_i\}_{i \in I}$ is locally finite, the set $J := \{i \in I : \mathcal{Z}(\mathfrak{q}_i) \cap K \neq \emptyset\}$ is finite. Let $\mathfrak{a}_1 := \bigcap_{i \in J} \mathfrak{q}_i$ and $\mathfrak{a}_2 := \bigcap_{i \notin J} \mathfrak{q}_i$; clearly, $\mathfrak{a} = \mathfrak{a}_1 \cap \mathfrak{a}_2$.

Since $K \subset X \setminus \bigcup_{i \notin J} \mathcal{Z}(\mathfrak{q}_i)$ and K is holomorphically convex, by [GR, VII.A.Prop.3] there is an Oka–Weil neighbourhood U of K in $X \setminus \bigcup_{i \notin J} \mathcal{Z}(\mathfrak{q}_i)$. Now, by [GR, VIII.A.Thm.11], there exist a holomorphic function $H \in \mathfrak{a}_2 = H^0(X, \mathfrak{a}_2\mathcal{O}_X)$ which is close to 1 on K . On the other hand, since $\mathcal{I}(\mathcal{Z}(\mathfrak{q}_i)) = \sqrt{\mathfrak{q}_i}$ for all i and there exists $m_i \geq 1$ such that $(\sqrt{\mathfrak{q}_i})^{m_i} \subset \mathfrak{p}_i$ (see Theorem 2.1), we find $m \geq 1$ such that $(\sqrt{\mathfrak{a}_1})^m \subset \mathfrak{a}_1$. Moreover, since J is a finite set

$$\begin{aligned} \mathcal{I}(\mathcal{Z}(\mathfrak{a})) &= \mathcal{I}(\mathcal{Z}(\mathfrak{a}_2 \cap \mathfrak{a}_1)) = \mathcal{I}\left(\mathcal{Z}(\mathfrak{a}_2) \cap \bigcap_{i \in J} \mathfrak{q}_i\right) = \mathcal{I}(\mathcal{Z}(\mathfrak{a}_2)) \cap \bigcap_{i \in J} \mathcal{I}(\mathcal{Z}(\mathfrak{q}_i)) \\ &= \mathcal{I}(\mathcal{Z}(\mathfrak{a}_2)) \cap \bigcap_{i \in J} \sqrt{\mathfrak{q}_i} = \mathcal{I}(\mathcal{Z}(\mathfrak{a}_2)) \cap \sqrt{\mathfrak{a}_1}. \end{aligned}$$

Thus, if $G \in \mathcal{I}(\mathcal{Z}(\mathfrak{a}))$, then $(HG)^m \in \mathfrak{a}_2\mathfrak{a}_1 \subset \mathfrak{a}_2 \cap \mathfrak{a}_1 = \mathfrak{a}$, that is, $HG \in \sqrt{\mathfrak{a}}$, and so $\mathcal{I}(\mathcal{Z}(\mathfrak{a})) \subset \mathfrak{R}_1(\mathfrak{a})$.

For the general case, we proceed as follows. By Lemma 2.3, we have that $\tilde{\mathfrak{a}} = \mathfrak{C}_1(\mathfrak{a}) \subset \mathfrak{C}_1(\sqrt{\mathfrak{a}}) = \mathfrak{R}_1(\mathfrak{a})$; hence, by Remarks 2.4

$$\mathcal{I}(\mathcal{Z}(\mathfrak{a})) = \mathcal{I}(\mathcal{Z}(\tilde{\mathfrak{a}})) = \mathfrak{R}_1(\tilde{\mathfrak{a}}) \subset \mathfrak{R}_1(\mathfrak{R}_1(\mathfrak{a})) = \mathfrak{R}_1(\mathfrak{a}) = \sqrt{\mathfrak{a}},$$

as wanted. \square

Remarks 2.5. (i) Observe that if \mathfrak{q} is a primary ideal of $\mathcal{O}(X)$, then, by Lemma 1.1,

$$\sqrt{\mathfrak{q}} = \begin{cases} \sqrt{\mathfrak{q}} & \text{if } \mathfrak{q} \text{ is saturated,} \\ H^0(X, \sqrt{\mathfrak{q}}\mathcal{O}_X) & \text{otherwise.} \end{cases}$$

(ii) There are saturated ideals \mathfrak{a} of $\mathcal{O}(X)$ whose radical $\sqrt{\mathfrak{a}}$ is not saturated. Consider the Stein space $(\mathbb{C}, \mathcal{O}_{\mathbb{C}})$ and for each $k \geq 1$ let $F, G \in \mathcal{O}(\mathbb{C})$ be holomorphic functions whose respective zero-sets are \mathbb{N} and such that $\text{mult}_n(F) = n$ and $\text{mult}_n(G) = 1$ for all $n \in \mathbb{N}$. Observe that the ideal \mathfrak{a} of $\mathcal{O}(\mathbb{C})$ generated by F is saturated because it is principal. However, its radical $\sqrt{\mathfrak{a}}$ is not saturated because $G \in \sqrt{\mathfrak{a}} \setminus \sqrt{\mathfrak{a}}$.

(iii) Conversely, there are non saturated ideals of $\mathcal{O}(X)$ whose radical $\sqrt{\mathfrak{a}}$ is saturated. Consider the Stein space $(\mathbb{C}, \mathcal{O}_{\mathbb{C}})$ and for each $k \geq 1$ let $F_k \in \mathcal{O}(\mathbb{C})$ be a holomorphic function whose zero-set is \mathbb{N} and such that

$$\text{mult}_n(F_k) := \begin{cases} 1 & \text{if } n < k, \\ 2 & \text{if } n \geq k. \end{cases}$$

Let \mathfrak{a} be the ideal of $\mathcal{O}(\mathbb{C})$ generated by the functions F_k . Let also $G \in \mathcal{O}(\mathbb{C})$ be a holomorphic function whose zero-set is \mathbb{N} and such that $\text{mult}_n(G) = 1$ for all $n \in \mathbb{N}$. Notice that $G^2 = F_1 \in \mathfrak{a}$ and $\sqrt{\mathfrak{a}} = G\mathcal{O}(\mathbb{C}) = \tilde{\mathfrak{a}} \neq \mathfrak{a}$.

3. THE REAL NULLSTELLENSATZ IN TERMS OF ŁOJASIEWICZ'S RADICAL

We present here some results relating Łojasiewicz's radical and the real radical in the abstract setting (see also [FG]).

3.1. The real radical in the abstract setting. We begin by recalling some properties concerning the classical Cauchy-Schwarz's inequality and Lagrange's equality. Cauchy-Schwarz's inequality says that in an Euclidean space $(E, \langle \cdot, \cdot \rangle)$ it holds that $|\langle x, y \rangle| \leq \|x\|\|y\|$, or equivalently, that $\langle x, y \rangle^2 \leq \|x\|^2\|y\|^2$ for every couple of vectors $x, y \in E$. For \mathbb{R}^n with its usual inner product we have:

$$(x_1y_1 + \cdots + x_ny_n)^2 \leq (x_1^2 + \cdots + x_n^2)(y_1^2 + \cdots + y_n^2) \quad \forall (x_1, \dots, x_n), (y_1, \dots, y_n) \in \mathbb{R}^n.$$

For instance, we can prove the previous inequality using the following polynomial identity in $\mathbb{Z}[\mathbf{x}, \mathbf{y}] := \mathbb{Z}[\mathbf{x}_1, \dots, \mathbf{x}_n; \mathbf{y}_1, \dots, \mathbf{y}_n]$, known as Lagrange's equality:

$$\begin{aligned} \left(\sum_{i=1}^n x_i^2 \right) \left(\sum_{j=1}^n y_j^2 \right) - \left(\sum_{k=1}^n x_k y_k \right)^2 &= \sum_{i,j=1}^n x_i^2 y_j^2 - \sum_{i,j=1}^n x_i y_i x_j y_j \\ &= \sum_{\substack{i,j=1 \\ i \neq j}}^n x_i^2 y_j^2 - 2 \sum_{\substack{i,j=1 \\ i < j}}^n x_i y_i x_j y_j = \sum_{\substack{i,j=1 \\ i < j}}^n (x_i y_j - x_j y_i)^2. \end{aligned} \quad (\text{LE})$$

Hence, if A is a (unitary commutative) ring and $a_1, \dots, a_n, b_1, \dots, b_n \in A$ it holds that

$$\left(\sum_{i=1}^n a_i^2\right)\left(\sum_{j=1}^n b_j^2\right) - \left(\sum_{k=1}^n a_k b_k\right)^2 = \sum_{\substack{i,j=1 \\ i < j}}^n (a_i b_j - a_j b_i)^2 \quad (\text{CS})$$

is a finite sum of squares. On the other hand, we say that an element $a \in A$ of a real ring A is non negative, and denoted by $a \geq 0$, if it belongs to all the prime cones of A . Using this facts, we prove the following result which presents the real radical in relation with Lojasiewicz's inequality.

Lemma 3.1. *Let A be a real ring and let \mathfrak{a} be an ideal of A . Then,*

$$\sqrt[n]{\mathfrak{a}} = \{a \in A : \exists b \in \mathfrak{a} \text{ \& } m \geq 1 \text{ such that } b - a^{2m} \geq 0\} \quad (3.4)$$

Moreover, if $\mathfrak{a} = (f_1, \dots, f_r)A$ and $f := f_1^2 + \dots + f_r^2$, then,

$$\sqrt[n]{\mathfrak{a}} = \{a \in A : \exists m \geq 1, \sigma \in \Sigma A^2 \text{ such that } \sigma f - a^{2m} \geq 0\}. \quad (3.5)$$

Proof. Denote by \mathfrak{b} the set in the right hand side of equality (3.4) and let us check $\sqrt[n]{\mathfrak{a}} = \mathfrak{b}$. First, take $a \in \sqrt[n]{\mathfrak{a}}$; there exists $a_1, \dots, a_r \in A$ and $m \geq 1$ such that

$$a^{2m} \leq a^{2m} + \sum_{i=1}^r a_i^2 =: b \in \mathfrak{a};$$

hence, $a \in \mathfrak{b}$.

Conversely, take now $a \in \mathfrak{b}$ and let $b \in \mathfrak{a}$ and $m \geq 1$ be such that $b - a^{2m} \geq 0$. Observe that there is no prime cone α in A such that $-b + a^{2m} \in \alpha$ and $b - a^{2m} \notin \text{supp}(\alpha)$. Thus, by the abstract Positivstellensatz (see [BCR, 4.4.1]), there exist sums of squares σ_1, σ_2 in A and a positive integer $\ell \geq 1$ such that $\sigma_1 + (-b + a^{2m})\sigma_2 + (-b + a^{2m})^{2\ell} = 0$. Therefore,

$$(-b + a^{2m})^{2\ell} + \sigma_1 + a^{2m}\sigma_2 = b\sigma_2 \in \mathfrak{a};$$

hence, $-b + a^{2m} \in \sqrt[n]{\mathfrak{a}}$ and since $b \in \mathfrak{a} \subset \sqrt[n]{\mathfrak{a}}$ and the latter is a radical ideal, we conclude that $a \in \sqrt[n]{\mathfrak{a}}$, as wanted.

Next if $\mathfrak{a} = (f_1, \dots, f_r)A$, it is clear that the set in the right hand side of equality (3.5) is contained in $\sqrt[n]{\mathfrak{a}}$. Conversely, let $a \in \sqrt[n]{\mathfrak{a}}$; there exists $b \in \mathfrak{a}$ and $\ell \geq 1$ such that $b - a^{2\ell} \geq 0$. Since $b \in \mathfrak{a}$ there exist $g_1, \dots, g_r \in A$ such that $b = g_1 f_1 + \dots + g_r f_r$. By 3.1(CS), we have that $b^2 \leq f\sigma$ where $\sigma = g_1^2 + \dots + g_r^2 \in \Sigma A^2$. On the other hand, since $b - a^{2\ell} \geq 0$, we have

$$b + a^{2\ell} = (b - a^{2\ell}) + 2a^{2\ell} \geq 0 \implies b^2 - a^{4\ell} = (b + a^{2\ell})(b - a^{2\ell}) \geq 0;$$

hence, if we denote $m := 2\ell$, we have $f\sigma - a^{2m} = (f\sigma - b^2) + (b^2 - a^{2m}) \geq 0$, as wanted. \square

3.2. Lojasiewicz's inequality and the real radical. Recall here the well-known facts that in the polynomial case and in the local analytic setting, Artin-Lang's Theorem relates the abstract positivity of an element in the corresponding ring with its geometric positivity. More precisely,

3.2.1. Polynomial case. Let R be a real closed field and let $X \subset R^n$ be an algebraic set. Denote by $R[X] := R[\mathbf{x}]/\mathcal{I}(X)$ the ring of polynomial functions on X , where $R[\mathbf{x}] := R[x_1, \dots, x_n]$ and $\mathcal{I}(X) = \{g \in R[\mathbf{x}] : g(x) = 0 \ \forall x \in X\}$. An element $f \in R[X]$ is ≥ 0 if and only if $f(x) \geq 0$ for all $x \in X$.

3.2.2. Local analytic case. Let $\mathcal{O}_n := \mathbb{R}\{\mathbf{x}\} := \mathbb{R}\{x_1, \dots, x_n\}$ and let $X_a \subset \mathbb{R}_a^n$ be an analytic germ at a point $a \in \mathbb{R}^n$. Denote by $\mathcal{O}(X_a) := \mathbb{R}\{\mathbf{x} - a\} / \mathcal{I}(X_a)$ the *ring of analytic function germs on X_a* , where $\mathcal{I}(X_a) := \{g_a \in \mathbb{R}\{\mathbf{x} - a\} : X_a \subset \mathcal{Z}(g_a)\}$. An element $f_a \in \mathcal{O}(X_a)$ is ≥ 0 if and only if there exist representatives X of X_a and f of f_a defined on X such that $f(x) \geq 0$ for all $x \in X$.

We recall below the well-known real Nullstellensätze in terms of the real radical.

Theorem 3.2 (Real Nullstellensatz). *Let A denote either $R[X]$ for an algebraic set X or $\mathcal{O}(X_a)$ for an analytic germ $X_a \subset \mathbb{R}_a^n$ and let \mathfrak{a} be an ideal of A . Then, $\mathcal{I}(\mathcal{Z}(\mathfrak{a})) = \sqrt[\mathbb{R}]{\mathfrak{a}}$.*

Now we use Łojasiewicz's inequality in order to prove that both in the ring of polynomials and in the ring of germs the real radical is the same as the Łojasiewicz's radical. Since both in algebraic and in the local analytic cases the geometric objects can be represented as the zero-set of a single positive semidefinite equation, it is enough to consider the cases $X := R^n$ and $X_a := \mathbb{R}_0^n$.

Lemma 3.3 (Łojasiewicz's inequality). *Let A denote either $R[\mathbf{x}]$ or \mathcal{O}_n and let $f, g \in A$ be such that $\mathcal{Z}(f) \subset \mathcal{Z}(g)$. Then, there exist integers $m, \ell \geq 0$ and a constant $C > 0$ such that $g^{2m} \leq C(1 + \|\mathbf{x}\|^2)^\ell |f|$. In particular, if $A = \mathcal{O}_n$ we can choose $\ell = 0$.*

For the proof of Łojasiewicz's inequality in the polynomial case use [BCR, 2.6.2 & 2.6.6] while for the local analytic case, we refer the reader to [BM, 6.4]. As a straightforward consequence of Łojasiewicz's inequality we obtain the following descriptions of the real radical in the geometric settings we are considering. Namely,

Corollary 3.4. *Let A denote either $R[X]$ for an algebraic set X or $\mathcal{O}(X_a)$ for an analytic germ $X_a \subset \mathbb{R}_a^n$. Let \mathfrak{a} be an ideal of A and let $f \in A$ be a positive semidefinite element such that $\mathcal{Z}(f) = \mathcal{Z}(\mathfrak{a})$. Then,*

$$\mathcal{I}(\mathcal{Z}(\mathfrak{a})) = \{g \in A : \exists m, \ell \geq 0, C > 0 \text{ such that } C(1 + \|\mathbf{x}\|^2)^\ell f - g^{2m} \geq 0\}.$$

In particular, if $A = \mathcal{O}(X_a)$ we can always choose $\ell = 0$.

4. REAL NULLSTELLENSATZ IN THE REAL ANALYTIC SETTING

Let $X \subset \mathbb{R}^n$ be a global analytic set endowed with its sheaf \mathcal{O}_X and let $\mathfrak{a} \subset \mathcal{O}(X)$ be an ideal. If \mathfrak{a} is finitely generated, say by $f_1, \dots, f_r \in \mathfrak{a}$, we have seen in Lemma 3.1 how to manage the function $f := \sum_{i=1}^r f_i^2$ in the definition of the Łojasiewicz radical, see (I.2). The following result provides an analogous tool for the case when \mathfrak{a} is not finitely generated.

Lemma 4.1 (Crespina Lemma). *Let \mathfrak{a} be an ideal of $\mathcal{O}(\mathbb{R}^n)$. Then, there exists $f \in \tilde{\mathfrak{a}}$ such that*

- (i) *f is an infinite sum of squares in $\mathcal{O}(\mathbb{R}^n)$ and $\mathcal{Z}(f) = \mathcal{Z}(\mathfrak{a})$.*
- (ii) *For each $g \in \tilde{\mathfrak{a}}$ there exists a unit $u \in \mathcal{O}(\mathbb{R}^n)$ such that $g^2 \leq fu$.*

Proof. By [C2, Prop.2 & 5] the sheaf of ideals $\mathfrak{a}\mathcal{O}_{\mathbb{R}^n}$ extends to a coherent invariant sheaf of ideals \mathcal{J} on a invariant open Stein neighborhood Ω of \mathbb{R}^n in \mathbb{C}^n . Let $\{L_k\}_{k \geq 1}$ be an exhaustion of Ω by compact sets. Since \mathcal{J} is invariant and coherent, we deduce, by Cartan's Theorem A (see [C2]), that there exists a countable collection of invariant holomorphic sections $\{G_\ell\}_{\ell \geq 1} \subset H^0(\Omega, \mathcal{J})$ such that for each $k \geq 1$ there exists $\ell(k)$ such that for

each $z \in L_k$ the germs $G_{1,z}, \dots, G_{\ell(k),z}$ generates the ideal \mathcal{I}_z . For each $k \geq 1$, define $\mu_k := \max_{L_k} \{|G_k|^2\} + 1$ and $\gamma_k := 1/\sqrt{2^k \mu_k}$; consider the series $H := \sum_{k \geq 1} \gamma_k^2 G_k^2$, which converges uniformly on the compact subsets of Ω . Indeed, let $L \subset \Omega$ be a compact set and observe that there exists an index $k_0 \geq 1$ such that $L \subset L_k$ for all $k \geq k_0$. Moreover, for each $z \in L$, we have that $\gamma_k^2 |G_k(z)|^2 \leq \frac{1}{2^k}$ and so

$$\left| \sum_{k \geq k_0} \gamma_k^2 G_k^2(z) \right| \leq \sum_{k \geq k_0} \gamma_k^2 |G_k(z)|^2 \leq \sum_{k \geq k_0} \frac{1}{2^k} \leq 1$$

for each $z \in L$. Denote $S_m := \sum_{k=1}^m \gamma_k^2 G_k^2 \in H^0(\Omega, \mathcal{I})$; hence, $F := \sum_{k \geq 1} \gamma_k^2 G_k^2 = \lim_{m \rightarrow \infty} S_m$ in the Frechet topology of $H^0(\Omega, \mathcal{I})$. Since $H^0(\Omega, \mathcal{I})$ is, by [C1, VIII.Thm.4, pag.60], a closed ideal of $H^0(\Omega, \mathcal{O}_\Omega)$, we conclude that $F \in H^0(\Omega, \mathcal{I})$ and so $f := F|_{\mathbb{R}^n} \in \tilde{\mathfrak{a}}$. For each $k \geq 1$, denote $f_k := (\gamma_k G_k)|_X$ and we write $f = \sum_{k \geq 1} f_k^2$.

Next, pick $g \in \tilde{\mathfrak{a}}$ and let $x \in \mathbb{R}^n$. By the choice of the G_k 's and since $g_x \in \mathfrak{a}\mathcal{O}_{\mathbb{R}^n, x}$, there exist $a_{1,x}, \dots, a_{r,x} \in \mathcal{O}_{\mathbb{R}^n, x}$ (with r depending on x) such that $g_x = a_{1,x}f_{1,x} + \dots + a_{r,x}f_{r,x}$; hence, using 3.1(CS)

$$g_x^2 \leq \left(\sum_{i=1}^r f_i^2 \right)_x \left(\sum_{i=1}^r a_{i,x}^2 \right) \leq f_x M_x$$

where M_x is a positive real number such that $\sum_{i=1}^r a_{i,x}^2 \leq M_x$.

Next, pick a compact set $K \subset \mathbb{R}^n$ and notice that we find a constant $M_K > 0$ such that $g^2|_K \leq f|_K M_K$. Now, fix an exhaustion $\{K_m\}_{m \geq 1}$ of X by compact sets and let $u \in \mathcal{O}(\mathbb{R}^n)$ be a strictly positive analytic function such that $M_{K_m} \leq u|_{K_m \setminus K_{m-1}}$ for all $m \geq 1$. A straightforward computation shows that $g^2 \leq fu$, as wanted. \square

Remark 4.2. Observe that in general $f \in \tilde{\mathfrak{a}} \setminus \mathfrak{a}$. Indeed, let $\mathfrak{a} \subset \mathcal{O}(X)$ be a proper ideal such that $\mathcal{Z}(\mathfrak{a}) = \emptyset$ (see for instance Example 1 in the Introduction). Then, there is no $f \in \mathfrak{a}$ such that $\mathcal{Z}(f) = \mathcal{Z}(\mathfrak{a})$ because otherwise $\mathfrak{a} = \mathcal{O}(X)$.

Proposition 4.3. *Let X be a global analytic set in \mathbb{R}^n and let $f, g \in \mathcal{O}(X)$ be analytic functions such that $\mathcal{Z}(f) \subset \mathcal{Z}(g)$. Let $K \subset X$ be a compact set. Then, there exist an integer $m \geq 1$ and an analytic function $h \in \mathcal{O}(X)$ such that $|h| < 1$, $\mathcal{Z}(h) \cap K = \emptyset$ and $|f| \geq (hg)^{2m}$.*

Proof. The proof of this result is entirely contained in [ABS]; for the sake of the reader we sketch the proof referring to the concrete statements in [ABS]. First, by [ABS, Cor. 2.3] there is a proper global analytic subset $Y_1 \subset Y := \mathcal{Z}(f)$ such that $K \cap Y_1 = \emptyset$, an integer m and an open neighborhood U of $Y \setminus Y_1$ contained in $X \setminus Y_1$ such that $g^{2m} < |f|$ on $U \setminus Y$.

Now, consider the global semianalytic set $S := \{x \in X : f^2 - g^{4m} < 0\}$ and its closure \overline{S} in X . Since U is open and $S \cap U = \emptyset$, we get $\overline{S} \cap U = \emptyset$; hence,

$$Y \cap \overline{S} \subset Y \setminus U \subset Y_1.$$

By means of [ABS, Thm. 2.5], we find a positive semidefinite equation h_0 of Y_1 such that $h_0 < |f|$ on $\overline{S} \setminus Y_1$ and $h_0 < 1$ in X . Thus, $(h_0 g)^{2m} \leq |f|$ on the open neighbourhood $V := \{g < 1\}$ of Y , because $h_0 < 1$. Finally, [ABS, Lem. 2.7] provides a positive unit $0 < h_1 < 1$ such that $(h_1 h_0)^{2m} g^{2m} < |f|$ on $\mathbb{R}^n \setminus Y$ and taking $h := h_0 h_1$, we are done. \square

Now, we are ready to prove Theorem 2.

Proof of Theorem 2. Following the notations of Definition 2.3, consider the ideals:

$$\mathfrak{L}_1(\mathfrak{a}) := \mathfrak{C}_1(\sqrt[l]{\mathfrak{a}}) \quad \& \quad \mathfrak{L}_2(\mathfrak{a}) := \mathfrak{C}_2(\sqrt[l]{\mathfrak{a}})$$

By Lemma 2.3 we have $\mathfrak{L}_1(\mathfrak{a}) = \mathfrak{L}_2(\mathfrak{a}) = \widetilde{\sqrt[l]{\mathfrak{a}}}$. Recall moreover that by Remark 2.4 (2), we have $\mathfrak{L}_i(\mathfrak{L}_i(\mathfrak{a})) = \mathfrak{L}_i(\mathfrak{a})$. We want to show that $\mathcal{I}(\mathcal{Z}(\mathfrak{a})) = \widetilde{\sqrt[l]{\mathfrak{a}}}$. Clearly $\widetilde{\sqrt[l]{\mathfrak{a}}} \subset \mathcal{I}(\mathcal{Z}(\mathfrak{a}))$, so it is enough to prove the inclusion $\mathcal{I}(\mathcal{Z}(\mathfrak{a})) \subset \mathfrak{L}_1(\mathfrak{a})$.

Assume first that \mathfrak{a} is a saturated ideal. By Lemma 4.1, there exists a positive semi-definite $f \in \mathfrak{a}$ such that $\mathcal{Z}(f) = \mathcal{Z}(\mathfrak{a})$. Let now $g \in \mathcal{I}(\mathcal{Z}(\mathfrak{a}))$ and let $K \subset X$ be a compact set. By Proposition 4.3, there exist an integer $m \geq 1$ and an analytic function $h \in \mathcal{O}(X)$ such that $\mathcal{Z}(h) \cap K = \emptyset$ and $f \geq (hg)^{2m}$, that is, $hg \in \sqrt[l]{\mathfrak{a}}$. Thus, $g \in \mathfrak{L}_1(\mathfrak{a})$ and so $\mathcal{I}(\mathcal{Z}(\mathfrak{a})) \subset \mathfrak{L}_1(\mathfrak{a})$.

For the general case, we proceed as follows. By Lemma 2.3, we have that $\tilde{\mathfrak{a}} = \mathfrak{C}_1(\mathfrak{a}) \subset \mathfrak{C}_1(\sqrt[l]{\mathfrak{a}}) = \mathfrak{L}_1(\mathfrak{a})$; hence,

$$\mathcal{I}(\mathcal{Z}(\mathfrak{a})) = \mathcal{I}(\mathcal{Z}(\tilde{\mathfrak{a}})) = \mathfrak{L}_1(\tilde{\mathfrak{a}}) = \mathfrak{L}_1(\mathfrak{C}_1(\mathfrak{a})) \subset \mathfrak{L}_1(\mathfrak{C}_1(\sqrt[l]{\mathfrak{a}})) = \mathfrak{L}_1(\mathfrak{L}_1(\mathfrak{a})) = \mathfrak{L}_1(\mathfrak{a}),$$

as wanted. \square

Remark 4.4. With the notations of the previous proof, if \mathfrak{a} is saturated and $f \in \mathfrak{a}$ satisfies that $\mathcal{Z}(\mathfrak{a}) = \mathcal{Z}(f)$, then

$$\mathcal{I}(\mathcal{Z}(\mathfrak{a})) = \mathcal{I}(\mathcal{Z}(f^2)) = \mathfrak{L}_1(f^2\mathcal{O}(X)) = \mathfrak{L}_2(f^2\mathcal{O}(X))$$

The previous equality can be understood as the counterpart of Lemma 3.1 and Corollary 3.4 in the global analytic setting.

4.1. Convex ideals. We say that an ideal \mathfrak{a} of $\mathcal{O}(X)$ is *convex* if each $g \in \mathcal{O}(X)$ satisfying $|g| \leq f$ for some $f \in \mathfrak{a}$ belongs to \mathfrak{a} . In particular, the Łojasiewicz's radical $\sqrt[l]{\mathfrak{a}}$ of an ideal \mathfrak{a} of $\mathcal{O}(X)$ is a radical convex ideal. Moreover, we define the *convex hull* $\mathfrak{g}(\mathfrak{a})$ of an ideal \mathfrak{a} of $\mathcal{O}(X)$ by

$$\mathfrak{g}(\mathfrak{a}) := \{g \in \mathcal{O}(X) : \exists f \in \mathfrak{a} \text{ such that } |g| \leq f\}$$

Notice that $\mathfrak{g}(\mathfrak{a})$ is the smallest convex ideal of $\mathcal{O}(X)$ that contains \mathfrak{a} and $\sqrt[l]{\mathfrak{a}} := \sqrt{\mathfrak{g}(\mathfrak{a})}$.

Remarks 4.5. (i) There are saturated ideals \mathfrak{a} of $\mathcal{O}(X)$ whose Łojasiewicz radical $\sqrt[l]{\mathfrak{a}}$ is not saturated and there are non saturated ideals of $\mathcal{O}(X)$ whose Łojasiewicz radical $\sqrt[l]{\mathfrak{a}}$ is saturated. To provide examples proceed similarly to the examples constructed in Remarks 2.5(ii) & (iii), but substituting \mathbb{C} by \mathbb{R} . Observe that in the examples we suggest, it holds $\sqrt[l]{\mathfrak{a}} = \sqrt{\mathfrak{a}}$.

(ii) There are convex saturated ideals which are not radical. Consider for instance the ideal $\mathfrak{a} := (x^2, xy, y^2)\mathcal{O}(\mathbb{R}^2)$ of $\mathcal{O}(\mathbb{R}^2)$.

(iii) There are radical saturated ideals which are not convex. Consider for instance the ideal $\mathfrak{a} := (x^2 + y^2)\mathcal{O}(\mathbb{R}^2)$ of $\mathcal{O}(\mathbb{R}^2)$.

(iv) There are convex radical ideals which are not saturated. Indeed, let $g_1, g_2 \in \mathcal{O}(\mathbb{R})$ be such that $\mathcal{Z}(g_1) = \mathcal{Z}(g_2) = \mathbb{N}$ and $\text{mult}_n(g_1) = n$ and $\text{mult}_n(g_2) = 1$ for all $n \geq 1$. Consider the analytic functions on \mathbb{R}^3 given by the formulae $f_i := z^2 + y^2 g_i(x)^2$. Notice that each f_i is an irreducible analytic function on \mathbb{R}^3 and $\mathcal{Z}(f_1) = \mathcal{Z}(f_2) = \mathcal{Z}(z, y) \cup \bigcup_{n \geq 1} \mathcal{Z}(z, x - n)$. Let $\mathfrak{a} := \sqrt[l]{f_1 \mathcal{O}(\mathbb{R}^3)}$ which is a radical convex ideal of $\mathcal{O}(\mathbb{R}^3)$; however, it is not saturated as we will see next. Otherwise, by Theorem 2, $\mathfrak{a} = \mathcal{I}(\mathcal{Z}(\mathfrak{a}))$ and so $f_2 \in \mathfrak{a}$; hence, there

exists $m \geq 1$ and $a \in \mathcal{O}(\mathbb{R}^3)$ such that $h := af_1 - f_2^{2m} \geq 0$. But this contradicts the fact that

$$\text{mult}_{\ell+1}(h) = \text{mult}_{\ell+1}(g_2) = 2\ell < 2\ell + 2 = \text{mult}_{4\ell+1}(g_1).$$

Therefore, \mathfrak{a} is not saturated.

Corollary 4.6. *Let $X \subset \mathbb{R}^n$ be a global analytic set and let \mathfrak{q} be a primary ideal of the ring $\mathcal{O}(X)$. Then, $\mathcal{I}(\mathcal{Z}(\mathfrak{q})) = \sqrt{\mathfrak{q}}$ if and only if \mathfrak{q} is saturated and $\sqrt{\mathfrak{q}}$ is convex. In particular, a saturated convex prime ideal satisfies $\mathcal{I}(\mathcal{Z}(\mathfrak{p})) = \mathfrak{p}$.*

Proof. The “only if” implication is clear. For the converse, observe first that $\mathcal{Z}(\mathfrak{q}) \neq \emptyset$ since \mathfrak{q} is primary and saturated. Let $g \in \mathcal{I}(\mathcal{Z}(\mathfrak{q})) = \mathfrak{L}_1(\mathfrak{q})$ (see the proof of Theorem 2) and let $x \in \mathcal{Z}(\mathfrak{q})$; then, there exists $m \geq 1$, $h \in \mathcal{O}(X)$ and $f \in \mathfrak{q}$ such that $h(x) \neq 0$ and $f \geq (hg)^{2m}$; in particular, $h \notin \sqrt{\mathfrak{q}}$. Now, since \mathfrak{q} is convex $(hg)^{2m} \in \mathfrak{q}$ and so $hg \in \sqrt{\mathfrak{q}}$; hence, $\sqrt{\mathfrak{q}}$ being prime, we deduce $g \in \sqrt{\mathfrak{q}}$. Therefore, $\mathcal{I}(\mathcal{Z}(\mathfrak{q})) \subset \sqrt{\mathfrak{q}}$ and so both ideals are equal. \square

Remarks 4.7. (i) Observe that if \mathfrak{a} is a convex ideal of the ring $\mathcal{O}(X)$, then so is $\sqrt{\mathfrak{a}}$. Indeed, let $f \in \sqrt{\mathfrak{a}}$ and $g \in \mathcal{O}(X)$ be such that $|g| \leq f$. Let $m \geq 1$ be such that $f^m \in \mathfrak{a}$; clearly, $|g^m| \leq f^m$. Thus, $g^m \in \mathfrak{a}$ and so $g \in \sqrt{\mathfrak{a}}$.

(ii) Consider the primary ideal $\mathfrak{q} := (x^2, y^2)\mathcal{O}(\mathbb{R}^2)$. Observe that $\sqrt{\mathfrak{q}} = (x, y)\mathcal{O}(\mathbb{R}^2) = \mathcal{I}(\mathcal{Z}(\mathfrak{q}))$; hence, $\sqrt{\mathfrak{q}}$ is convex. On the other hand, the functions $f := x^2 + y^2 \in \mathfrak{q}$ and $g := xy \in \mathcal{O}(\mathbb{R}^2)$ satisfy $|g| \leq f$ but $g \notin \mathfrak{q}$; hence, \mathfrak{q} is not convex.

(iii) Let \mathfrak{a} be a convex saturated ideal and let $\mathfrak{a} = \bigcap_{i \in I} \mathfrak{q}_i$ be a normal primary decomposition of \mathfrak{a} . Let $i_0 \in I$ be such that $\sqrt{\mathfrak{q}_{i_0}}$ is a minimal ideal of the family $\{\sqrt{\mathfrak{q}_i}\}_{i \in I}$. Then, \mathfrak{q}_{i_0} is a convex saturated prime ideal.

Proof. First, let us see that there exists $h_{i_0} \in \bigcap_{j \neq i_0} \mathfrak{q}_j \setminus \sqrt{\mathfrak{q}_{i_0}}$. Otherwise, by Lemma 1.2 there exists $j \neq i_0$ such that $\mathfrak{q}_i \subset \mathfrak{q}_{i_0}$; hence, $\sqrt{\mathfrak{q}_j} \subset \sqrt{\mathfrak{q}_{i_0}}$ and by the minimality of $\sqrt{\mathfrak{q}_{i_0}}$, we deduce that $\sqrt{\mathfrak{q}_j} = \sqrt{\mathfrak{q}_{i_0}}$, which contradicts the fact that the primary decomposition is normal. Thus, fix $h_{i_0} \in \bigcap_{j \neq i_0} \mathfrak{q}_j \setminus \sqrt{\mathfrak{q}_{i_0}}$.

Next, let $g \in \mathcal{O}(X)$ be such that $|g| \leq f$ for some $f \in \sqrt{\mathfrak{q}_{i_0}}$. Then, $|gh_{i_0}^2| \leq fh_{i_0}^2$ and since \mathfrak{a} is convex and $fh_{i_0}^2 \in \bigcap_{i \in I} \mathfrak{q}_i = \mathfrak{a}$, we deduce that $gh_{i_0}^2 \in \mathfrak{a} \subset \mathfrak{q}_{i_0}$; hence, we conclude that $g \in \mathfrak{q}_{i_0}$ because \mathfrak{q}_{i_0} is primary and $h_{i_0} \notin \sqrt{\mathfrak{q}_{i_0}}$. \square

(iv) Under the hypotheses of Remark 4.7(iii), the result is no longer true in general if $\sqrt{\mathfrak{q}_{i_0}}$ is not a minimal ideal of the family $\{\sqrt{\mathfrak{q}_i}\}_{i \in I}$. Indeed, let $\mathfrak{a} := \mathfrak{q}_1 \cap \mathfrak{q}_2 = (z^3(x^2 + y^2), z^4)\mathcal{O}(\mathbb{R}^3)$ be the “normal” intersection of the primary ideals $\mathfrak{q}_1 := z^3\mathcal{O}(\mathbb{R}^3)$ and $\mathfrak{q}_2 := (x^2 + y^2, z^4)\mathcal{O}(\mathbb{R}^3)$. Let us check that \mathfrak{a} is convex. Indeed, if $f \in \mathfrak{a}$ is positive semidefinite, then z^4 divides f . Now, a straightforward computation shows that if $g \in \mathcal{O}(\mathbb{R}^3)$ satisfies $|g| \leq f$, then z^4 divides g and so $g \in \mathfrak{a}$; hence, \mathfrak{a} is convex. However, \mathfrak{q}_2 is not convex because $x^2 \leq x^2 + y^2$ but $x^2 \notin \mathfrak{q}_2$. Observe that $\sqrt{\mathfrak{q}_1} \subsetneq \sqrt{\mathfrak{q}_2}$.

As we have already seen in Theorem 2 the equality $\mathcal{I}(\mathcal{Z}(\mathfrak{a})) = \widetilde{\sqrt{\mathfrak{g}(\mathfrak{a})}}$ holds for each ideal \mathfrak{a} of $\mathcal{O}(X)$, where $X \subset \mathbb{R}^n$ is a global analytic set. The last part of this section will be dedicated to determine until what extent we can modificate the order of applying the operations \sim , $\sqrt{\cdot}$ and $\mathfrak{g}(\cdot)$.

Lemma 4.8. *Let $X \subset \mathbb{R}^n$ be a global analytic set and let \mathfrak{a} be an ideal of $\mathcal{O}(X)$. Then,*

- (i) If \mathfrak{a} is convex, then $\tilde{\mathfrak{a}}$ is also convex.
- (ii) If \mathfrak{a} is saturated, there exists $f \in \mathfrak{a}$ such that $(\widetilde{\mathfrak{g}(\mathfrak{a})})^2 \subset \mathfrak{g}(f\mathcal{O}(X)) \subset \mathfrak{g}(\mathfrak{a})$.

Proof. We begin by proving (i), let $g \in \mathcal{O}(X)$ and $f \in \tilde{\mathfrak{a}}$ be such that $|g| \leq f$. Let $K \subset X$ be a compact set; by Lemma 2.3, there exists $h \in \mathcal{O}(X)$ such that $\mathcal{Z}(h) \cap K = \emptyset$ and $h^2 f \in \mathfrak{a}$. Since $|h^2 g| \leq h^2 f$ and \mathfrak{a} is convex, we deduce that $h^2 g \in \mathfrak{a}$; hence, by Lemma 2.3, we deduce that $g \in \tilde{\mathfrak{a}}$ and so $\tilde{\mathfrak{a}}$ is convex.

Next, we prove (ii). Since $\mathcal{O}(X) = \mathcal{O}(\mathbb{R}^n)/\mathcal{I}(X)$ by the correspondence theorem for ideals, there exists an ideal \mathfrak{b} of $\mathcal{O}(\mathbb{R}^n)$ which contains $\mathcal{I}(X)$ such that $\mathfrak{a} = \mathfrak{b}/\mathcal{I}(X)$. To be clearer in this part of the proof, we denote by $\hat{h} := h + \mathcal{I}(X)$ the elements of $\mathcal{O}(X) = \mathcal{O}(\mathbb{R}^n)/\mathcal{I}(X)$. By Lemma 4.1, there is $f \in \tilde{\mathfrak{b}}$ such that for each $a \in \tilde{\mathfrak{b}}$ there exist a unit $u \in \mathcal{O}(\mathbb{R}^n)$ satisfying $a^2 \leq fu$; hence, $\hat{a}^2 \leq \hat{f}\hat{u}$.

Pick $\hat{g} \in \widetilde{\mathfrak{g}(\mathfrak{a})}$ and let $K \subset X$ be a compact set. Then, there exists a function $\hat{h}_K \in \mathcal{O}(X)$ such that $\mathcal{Z}(\hat{h}_K) \cap K = \emptyset$ and $\hat{h}_K \hat{g} \in \mathfrak{g}(\mathfrak{a})$; hence, there is $\hat{a}_K \in \mathfrak{a}$ such that $|\hat{h}_K \hat{g}| \leq \hat{a}_K$.

Let $u_K \in \mathcal{O}(\mathbb{R}^n)$ be a unit such that $a_K^2 \leq fu_K$ and let $M_K > 0$ be such that $\hat{g}^2|_K \leq \hat{f}|_K M_K$ (recall that $|\hat{h}_K \hat{g}| \leq \hat{a}_K$ and $\mathcal{Z}(\hat{h}_K) \cap K = \emptyset$). Now, fix an exhaustion $\{K_m\}_{m \geq 1}$ of X by compact sets and let $\hat{u} \in \mathcal{O}(X)$ be a strictly positive analytic function such that $M_{K_m} \leq \hat{u}|_{K_m \setminus K_{m-1}}$ for all $m \geq 1$. A straightforward computation shows that $\hat{g}^2 \leq \hat{f}\hat{u}$; hence, $\hat{g}^2 \in \mathfrak{g}(\hat{f}\mathcal{O}(X))$.

To finish observe that if $\hat{g}_1, \hat{g}_2 \in \widetilde{\mathfrak{g}(\mathfrak{a})}$, there exists strictly positive analytic functions $\hat{u}_1, \hat{u}_2 \in \mathcal{O}(X)$ such that $\hat{g}_i^2 \leq \hat{f}\hat{u}_i^2$ for $i = 1, 2$; hence, $|\hat{g}_1 \hat{g}_2| \leq \hat{g}_1^2 + \hat{g}_2^2 \leq \hat{f}(\hat{u}_1^2 + \hat{u}_2^2)$ and so $\hat{g}_1 \hat{g}_2 \in \mathfrak{g}(\hat{f}\mathcal{O}(X))$. Thus, $(\widetilde{\mathfrak{g}(\mathfrak{a})})^2 \subset \mathfrak{g}(\hat{f}\mathcal{O}(X))$, as wanted. \square

Remark 4.9. (i) Observe that if \mathfrak{a} is saturated, then $\sqrt{\widetilde{\mathfrak{g}(\mathfrak{a})}} = \sqrt{\mathfrak{g}(h\mathcal{O}(X))} = \sqrt{\mathfrak{g}(\mathfrak{a})}$.

(ii) It seems plausible that if \mathfrak{a} is saturated, also $\mathfrak{g}(\mathfrak{a})$ is saturated, but till the moment we have not been able to prove it.

(iii) Suppose now that \mathfrak{a} is a convex saturated ideal of $\mathcal{O}(X)$ and $\mathfrak{a} = \bigcap_{i \in I} \mathfrak{q}_i$ is a normal primary decomposition of \mathfrak{a} . Let J be the set of indices $j \in I$ such that $\sqrt{\mathfrak{q}_j}$ is a minimal element of the family $\{\sqrt{\mathfrak{q}_i}\}_{i \in I}$; by Remark 4.7(iii), each primary ideal \mathfrak{q}_j is convex. It holds that

$$\widetilde{\sqrt{\mathfrak{a}}} = \widetilde{\bigvee \mathfrak{a}} = \mathcal{I}(\mathcal{Z}(\mathfrak{a})) = \bigcap_{j \in J} \sqrt{\mathfrak{q}_j} = \bigcap_{i \in I} \sqrt{\mathfrak{q}_i}.$$

(iv) If we are working in the framework of convex saturated ideals it holds an analogous result to Theorem 2.2 just changing “Stein space” by “global analytic set” and “closed ideal” by “convex saturated ideal”. As one can expect, the proof runs analogously to the one of Theorem 2.2 (see [F, §5.Satz 9]) and we left the concrete details to the reader.

5. THE REAL ANALYTIC RADICAL AND THE REAL NULLSTELLENSATZ

In this section we prove Theorem 3, that is, we relate the real Nullstellensatz with the classical real radical by means of the representation of positive semidefinite functions as sums of squares of meromorphic functions. We begin by recalling the definition of H -sets and H^a -sets and presenting some properties.

Definition 5.1. A global analytic set $Z \subset \mathbb{R}^n$ is an H -set if each positive semidefinite analytic function $f \in \mathcal{O}(\mathbb{R}^n)$ whose zero-set is Z , can be represented as a sum of squares of meromorphic functions on \mathbb{R}^n . More generally, we say that Z is an H^a -set if such representation may involve infinitely many squares.

The following properties are stated and proved for H^a -sets but many of them work analogously for H -sets.

Remarks 5.2. (i) Let $Y \subset Z \subset \mathbb{R}^n$ be global analytic sets. If Z is an H^a -set, then Y is also an H^a -set.

Indeed, let $f \in \mathcal{O}(\mathbb{R}^n)$ be a positive semidefinite analytic function such that $\mathcal{Z}(f) = Y$. Let now $g \in \mathcal{O}(\mathbb{R}^n)$ be an analytic function such that $\mathcal{Z}(g) = Z$. Observe that $h := g^2 f$ is positive semidefinite and $\mathcal{Z}(h) = Z$; hence, h is a sum of squares of meromorphic functions on \mathbb{R}^n and so the same happens for f . Thus, Y is an H^a -set.

(ii) In particular, if $Z \subset \mathbb{R}^n$ is an H -set, the same holds for each global irreducible component of Z .

(iii) Let $Z \subset \mathbb{R}^n$ be a global analytic set. Then, Z is an H^a -set if and only if there exists a positive semidefinite $f \in \mathcal{O}(\mathbb{R}^n)$ such that $\mathcal{Z}(f) = Z$ and each $g \in \mathcal{O}(\mathbb{R}^n)$ with $\mathcal{Z}(g) = Z$ and $0 \leq g \leq f$ is a sum of squares of meromorphic functions on \mathbb{R}^n .

Proof. The “only if” implication is clear. Conversely, assume that there exists a positive semidefinite analytic equation f of Z with the property in the statement and let $g \in \mathcal{O}(\mathbb{R}^n)$ be another positive equation of Z . Observe that

$$f - \left(\frac{f}{\sqrt{1+fg}} \right)^2 g = f \left(1 - \frac{fg}{1+fg} \right) \geq 0 \quad \& \quad \mathcal{Z} \left(\left(\frac{f}{\sqrt{1+fg}} \right)^2 g \right) = Z$$

Thus, $h := \left(\frac{f}{\sqrt{1+fg}} \right)^2 g$, and so g , is a sum of squares of meromorphic functions on \mathbb{R}^n ; hence, Z is an H^a -set. \square

(iv) By [Jw, Rz] each compact global analytic subset of \mathbb{R}^n is an H -set. Therefore, by [ABFR3, 1.9] each global analytic set Z whose global irreducible components are compact is an H^a -set.

(v) Let Z be a global analytic set. As a straightforward consequence of [ABF, 1.2], one deduces that Z is an H^a -set if and only if each global irreducible function $f \in \mathcal{O}(\mathbb{R}^n)$ with $\mathcal{Z}(f) \subset Z$ is a sum of squares of meromorphic functions on \mathbb{R}^n .

(vi) Note that Hilbert’s 17th Problem in its more general formulation involving infinite sums of squares has a positive answer for $\mathcal{O}(\mathbb{R}^n)$ if and only if all the global analytic subsets of \mathbb{R}^n of dimensions $1 \leq d \leq n-2$ are H^a -sets.

Next lemma will be used in the proof of Theorem 3 to move suitably the complex zero-sets of denominators arising in the representation of positive semidefinite analytic functions as sums of squares of meromorphic functions.

Lemma 5.3 (Perturbing denominators). *Let $f, b \in \mathcal{O}(\mathbb{R}^n)$ be non constant analytic functions and let Ω be an invariant neighbourhood of \mathbb{R}^n in \mathbb{C}^n on which f have a holomorphic extension F . Let $Z \subset \Omega$ be a complex analytic set such that $Z_{x_0} \not\subset \mathcal{Z}(F)_{x_0}$ for some $x_0 \in \mathcal{Z}(f)$. Then, there exist an analytic diffeomorphism $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that:*

- (i) $f \circ \varphi = fu$ for some unit $u \in \mathcal{O}(\mathbb{R}^n)$.

- (ii) $Z_{x_0} \not\subset \mathcal{Z}(B_0)_{x_0}$, where $B_0 : \Omega_0 \rightarrow \mathbb{C}$ is the holomorphic extension of $b_0 := b \circ \varphi$ to a small enough open neighbourhood $\Omega_0 \subset \Omega$ of \mathbb{R}^n in \mathbb{C}^n .

Proof. We may assume that b extends to a holomorphic function B on Ω and that $Z_{x_0} \subset \mathcal{Z}(B)_{x_0}$ because otherwise taking $\varphi = \text{id}$, we are done.

Fix a strictly positive analytic function $\varepsilon \in \mathcal{O}(\mathbb{R}^n)$ and for each $\lambda := (\lambda_1, \dots, \lambda_n) \in [-1, 1]^n$, consider the analytic map

$$\phi_\lambda : \mathbb{R}^n \rightarrow \mathbb{R}^n, \quad x \mapsto x + f^2(x)\varepsilon(x)\lambda.$$

We choose ε small enough, in such a way that ϕ_λ is, by [H, 2.1.7], an analytic diffeomorphism for each $\lambda \in [-1, 1]^n$. Since the function

$$f_0 : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}, \quad (x, y, t) \mapsto f(x + ty) - f(x)$$

vanishes identically on the set $\mathbb{R}^n \times \mathbb{R}^n \times \{0\}$, there is an analytic $h \in \mathcal{O}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R})$ such that $f_0 = ht$. Thus,

$$f \circ \phi_\lambda(x) = f(x) + f(x)^2\varepsilon(x)h(x, \lambda, f(x)^2\varepsilon(x)) = f(x)u_\lambda(x) \quad (5.6)$$

where $u_\lambda(x) := 1 + f(x)\varepsilon(x)h(x, \lambda, f(x)^2\varepsilon(x))$. Note also that $\mathcal{Z}(f \circ \phi_\lambda) \subset \mathcal{Z}(f)$.

Indeed, if $x \in \mathbb{R}^n$ satisfies $f \circ \phi_\lambda(x) = 0$, then $y := \phi_\lambda(x) \in \mathcal{Z}(f)$. Since ϕ_λ is bijective and $\phi_\lambda(y) = y$ (because $f(y) = 0$), we deduce that $x = y \in \mathcal{Z}(f)$.

Thus, since by its very definition u_λ is a unit on a neighbourhood of $\mathcal{Z}(f)$ and it does not vanish outside $\mathcal{Z}(f) = \mathcal{Z}(f \circ \phi_\lambda)$ (see (5.6)), we conclude that u_λ is a unit in $\mathcal{O}(\mathbb{R}^n)$ for all $\lambda \in [-1, 1]^n$. Therefore, the diffeomorphisms ϕ_λ satisfy condition (i) for all $\lambda \in [-1, 1]^n$.

Let us find now $\lambda_0 \in [-1, 1]^n$ such that $\varphi := \phi_{\lambda_0}$ satisfies also condition (ii). Consider the family of diffeomorphisms ϕ_λ as the analytic map

$$\phi : \mathbb{R}^n \times [-1, 1]^n \rightarrow \mathbb{R}^n, \quad (x, \lambda) \mapsto \phi_\lambda(x)$$

After shrinking Ω , we may assume that ε, b extend holomorphically to $E, B \in \mathcal{O}(\Omega)$; we can also assume that Ω is connected. Thus, ϕ extends to the holomorphic map

$$\Phi : \Omega \times \mathbb{C}^n \rightarrow \mathbb{C}^n, \quad (z, \mu) \mapsto z + F^2(z)E(z)\mu.$$

Let $U := \Phi^{-1}(\Omega)$ and consider the holomorphic function

$$B \circ \Phi : U \rightarrow \mathbb{C}, \quad (w, \mu) \mapsto B \circ \Phi(w, \mu)$$

Fix a polydisc $\Delta_0 \times \Delta_1 \subset \Omega \times \mathbb{C}^n$ of center $(x_0, 0)$ and radius $0 < \rho < 1$ contained in U :

(5.3.1) *The map $(B \circ \Phi)_w : \Delta_1 \rightarrow \mathbb{C}$, $\mu \mapsto (B \circ \Phi)(w, \mu)$ is not identically zero for each $w \in \Delta_0$.*

Otherwise, there exists $w \in \Delta_0$ such that

$$(B \circ \Phi)_w(\mu) := B \circ \Phi(w, \mu) = B(w + F^2(w)E(w)\mu)$$

is identically zero on the polydisc Δ_1 ; by the Identity Principle, we deduce that B is identically zero, which contradicts the hypothesis that b is non constant.

Next, since $Z_{x_0} \not\subset \mathcal{Z}(F)_{x_0}$, there exists, by the complex curve selection lemma, a complex analytic curve $\gamma : \mathbb{D}_\delta \rightarrow Z$ (defined on the disk \mathbb{D}_δ) such that $\gamma(\mathbb{D}_\delta) \subset \Delta_0$, $\gamma(0) = x_0$ and $\gamma(s) \notin \mathcal{Z}(F)$ for all $s \neq 0$. Consider the holomorphic function

$$G : \mathbb{D}_\delta \times \Delta_1 \rightarrow \mathbb{C}, \quad (s, \mu) \mapsto (B \circ \Phi)(\gamma(s), \mu).$$

We know, by 5.3.1, that the holomorphic function $G_s : \Delta_1 \rightarrow \mathbb{C}$, $\mu \mapsto G(s, \mu)$ is not identically zero for each $s \in \mathbb{D}_\delta$. Choose now a sequence $\{s_k\}_k \subset \mathbb{D}_\delta$ converging to 0 and observe that for each k the set $W_k := (\Delta_1 \cap \mathbb{R}^n) \setminus \mathcal{Z}(G_{s_k}) = [-\rho, \rho]^n \setminus \mathcal{Z}(G_{s_k})$ is open and dense in $\Delta_1 \cap \mathbb{R}^n = [-\rho, \rho]^n$ because each $\mathcal{Z}(G_{s_k})$ is a proper analytic subset of Δ_1 . By Baire's Theorem, the intersection $W := \bigcap_{k \geq 1} W_k$ is dense in $\Delta_1 \cap \mathbb{R}^n$ and we choose $\lambda_0 \in W$.

Now, observe that if $b_0 := b \circ \phi_{\lambda_0}$, then $B_0 := B \circ \Phi_{\lambda_0}$ is its holomorphic extension to Ω , where $\Phi_{\lambda_0} : \Omega \rightarrow \mathbb{C}^n$, $z \mapsto \Phi(z, \lambda_0)$. By the choice of λ_0 , we have that $B_0 \circ \gamma(s_k) \neq 0$ for all $k \geq 1$; hence $B_0 \circ \gamma$ is not identically zero on \mathbb{D}_δ and so the germ $(B_0 \circ \gamma)_0 \neq 0$. We conclude that $Z_{x_0} \not\subset \mathcal{Z}(B_0)_{x_0}$, as wanted. \square

Once this is proved we approach the proof of Theorem 3. We prove it when $\mathcal{Z}(\mathfrak{a})$ is an H^a -set, but the proof runs analogously if $\mathcal{Z}(\mathfrak{a})$ is an H -set.

Proof of Theorem 3. The proof is approached in several steps:

STEP 1. Assume first that $\mathfrak{a} = \mathfrak{p}$ is a prime ideal. Since $\mathcal{O}(X) = \mathcal{O}(\mathbb{R}^n)/\mathcal{I}(X)$, we may assume by means of the correspondence theorem for ideals that \mathfrak{p} is a saturated real prime ideal of $\mathcal{O}(\mathbb{R}^n)$. Observe that the “only if” implication is clear since $\mathcal{I}(\mathcal{Z}(\mathfrak{p}))$ is real analytic and saturated. For the converse, we proceed as follows. By [C2, Prop.2 & 5] the sheaf of ideals $\mathfrak{p}\mathcal{O}_{\mathbb{R}^n}$ extends to a coherent sheaf of ideals \mathcal{J} on an invariant open Stein neighborhood Ω of \mathbb{R}^n in \mathbb{C}^n . Since \mathfrak{p} is saturated $\mathfrak{p} = H^0(\mathbb{R}^n, \mathfrak{p})$; denote by $Z := \{z \in \Omega : \mathcal{J}_z \neq \mathcal{O}_{\mathbb{C}^n, z}\}$ the support in Ω of \mathcal{J} . Suppose, by way of contradiction, that there exists a function $g \in \mathcal{I}(\mathcal{Z}(\mathfrak{p})) \setminus \mathfrak{p}$. After shrinking Ω if necessary, we may assume that g extends to a holomorphic function G on Ω . Since $g \notin \mathfrak{p}$, there exists a point $x_0 \in \mathbb{R}^n$ such that $Z_{x_0} \not\subset \mathcal{Z}(G)_{x_0}$ but $\mathcal{Z}(\mathfrak{p}) \subset \mathcal{Z}(g)$, where $g := G|_{\mathbb{R}^n}$. By Proposition 4.3, there exists $f \in \mathfrak{p}$, $h \in \mathcal{O}(\mathbb{R}^n)$ and $m \geq 1$ such that $h(x_0) \neq 0$ and $f_1 := f - h^2 g^{2m} \geq 0$; clearly, since $h(x_0) \neq 0$, we have $h \notin \mathfrak{p}$. Moreover, since $\mathcal{Z}(f) \subset \mathcal{Z}(g)$, we may assume that $\mathcal{Z}(f_1) = \mathcal{Z}(f)$ just taking $\frac{h}{\sqrt{1+h^2 g^{2m}}}$ instead of h ; indeed,

$$\mathcal{Z}(f_1) = \mathcal{Z}((f - h^2 g^{2m}) + (f h^2 g^{2m})) = \mathcal{Z}(f - h^2 g^{2m}) \cap \mathcal{Z}(f h^2 g^{2m}) = \mathcal{Z}(f) \cap \mathcal{Z}(h g) = \mathcal{Z}(f).$$

Observe also that after shrinking Ω if necessary, f_1 extends to a holomorphic function $F_1 : \Omega \rightarrow \mathbb{C}$ such that $Z_{x_0} \not\subset \mathcal{Z}(F_1)_{x_0}$, because otherwise $Z_{x_0} \subset \mathcal{Z}(G)_{x_0}$, a contradiction.

Since $\mathcal{Z}(\mathfrak{p}) = \mathcal{Z}(f_1)$ is an H^a -set, there exists a non identically zero $b \in \mathcal{O}(\mathbb{R}^n)$ such that $b^2 f_1 = \sum_{i \geq 1} a_i^2$ for some $a_i \in \mathcal{O}(\mathbb{R}^n)$. By Lemma 5.3, there exist an analytic diffeomorphism $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that:

- (i) $f_1 \circ \varphi = f_1 u$ for some unit $u \in \mathcal{O}(\mathbb{R}^n)$.
- (ii) $Z_{x_0} \not\subset \mathcal{Z}(B_1)_{x_0}$, where $B_1 : \Omega_0 \rightarrow \mathbb{C}$ is the holomorphic extension of $b_1 := b \circ \varphi$ to a small enough open neighbourhood $\Omega_0 \subset \Omega$ of \mathbb{R}^n in \mathbb{C}^n .

Let $v \in \mathcal{O}(\mathbb{R}^n)$ be a strictly positive unit such that $v^2 = u^{-1}$; hence,

$$b_1^2 f = b_1^2 h^2 g^{2m} + b_1^2 f_1 = b_1^2 h^2 g^{2m} + \sum_{i \geq 1} ((a_i \circ \varphi) v)^2.$$

Observe that since $Z_{x_0} \not\subset \mathcal{Z}(B_1)_{x_0}$, we have $b_1 \notin \mathfrak{p}$. Since $f \in \mathfrak{p}$ and \mathfrak{p} is a real ideal (resp. ∞ -real ideal), we deduce that $b_1 h g^m \in \mathfrak{p}$, which contradicts the fact that $b_1, h, g \notin \mathfrak{p}$. Thus, we conclude that $\mathcal{I}(\mathcal{Z}(\mathfrak{p})) = \mathfrak{p}$, as wanted.

STEP 2. Next assume that \mathfrak{a} is a saturated real analytic ideal of $\mathcal{O}(X)$ whose zero-set is an H^a -set. By Proposition 1.3 and Corollary 1.4 \mathfrak{a} admits a normal primary decomposition $\mathfrak{a} = \bigcap_i \mathfrak{q}_i$, such that all the ideals \mathfrak{q}_i are saturated real analytic prime ideals. Since $\mathcal{Z}(\mathfrak{a}) = \bigcup_i \mathcal{Z}(\mathfrak{q}_i)$ is an H^a -set, we deduce by Remark 5.2(i) that each $\mathcal{Z}(\mathfrak{q}_i)$ is an H^a -set. Now, by Step 1, $\mathcal{I}(\mathcal{Z}(\mathfrak{q}_i)) = \mathfrak{q}_i$ for each i . Thus,

$$\mathcal{I}(\mathcal{Z}(\mathfrak{a})) = \mathcal{I}\left(\bigcup_{i \in I} \mathcal{Z}(\mathfrak{q}_i)\right) = \bigcap_{i \in I} \mathcal{I}(\mathcal{Z}(\mathfrak{q}_i)) = \bigcap_{i \in I} \mathfrak{q}_i = \mathfrak{a}.$$

STEP 3. We approach now the general case, that is, \mathfrak{a} is an ideal of $\mathcal{O}(X)$, whose zero-set is an H^a -set. Since $\mathcal{I}(\mathcal{Z}(\mathfrak{a})) = \mathcal{I}(\mathcal{Z}(\widetilde{\sqrt[a]{\mathfrak{a}}}))$, it is enough to check, in view of the previous case, that $\widetilde{\sqrt[a]{\mathfrak{a}}}$ is a real analytic ideal. Indeed, let $\sum_{k \geq 1} a_k^2 \in \widetilde{\sqrt[a]{\mathfrak{a}}}$ and let $K \subset X$ be a compact set; by Lemma 2.3, there exists $h \in \mathcal{O}(X)$ such that $\mathcal{Z}(h) \cap K = \emptyset$ and $h \sum_{k \geq 1} a_k^2 \in \sqrt[a]{\mathfrak{a}}$; hence, $\sum_{k \geq 1} (ha_k)^2 \in \sqrt[a]{\mathfrak{a}}$. Since $\sqrt[a]{\mathfrak{a}}$ is real analytic, we deduce that each $ha_k \in \sqrt[a]{\mathfrak{a}}$. Since this happens for all compact set $K \subset X$, we deduce by Lemma 2.3 that each $a_k \in \widetilde{\sqrt[a]{\mathfrak{a}}}$. Thus, $\widetilde{\sqrt[a]{\mathfrak{a}}}$ is a real analytic ideal, and we are done. \square

Remarks 5.4. Let $\mathfrak{a} \subset \mathcal{O}(X)$ be an ideal. Then,

(i) $\mathfrak{a} \subset \sqrt[a]{\mathfrak{a}} \subset \sqrt[b]{\mathfrak{a}}$.

(ii) If $\mathcal{Z}(\mathfrak{a})$ is H^a -set, we have moreover

- $\widetilde{\sqrt[a]{\mathfrak{a}}} = \sqrt[b]{\mathfrak{a}} = \mathcal{I}(\mathcal{Z}(\mathfrak{a}))$.
- For each $g \in \sqrt[b]{\mathfrak{a}}$ there exists $b \in \mathcal{O}(X)$ such that $\mathcal{Z}(b) \subset \mathcal{Z}(g)$ and $bg \in \sqrt[a]{\mathfrak{a}}$.

Indeed, let $g \in \sqrt[b]{\mathfrak{a}}$; then, there exists $f \in \mathfrak{a}$ and $m \geq 1$ such that $f - g^{2m} \geq 0$; in particular, $f \geq 0$. Observe that $\mathcal{Z}(f) \subset \mathcal{Z}(g)$ and taking $f' := 2f \in \mathfrak{a}$ instead of f , we may assume that $\mathcal{Z}(f) = \mathcal{Z}(f - g^{2m})$; indeed,

$$\mathcal{Z}(f' - g^{2m}) = \mathcal{Z}(f + (f - g^{2m})) = \mathcal{Z}(f) \cap \mathcal{Z}(f - g^{2m}) = \mathcal{Z}(f) \cap \mathcal{Z}(g^{2m}) = \mathcal{Z}(f')$$

Now, since $\mathcal{Z}(\mathfrak{a})$ is an H^a -set, we deduce, by [ABFR3, 4.1], that there exist $m \geq 1$ and $b, a_k \in \mathcal{O}(X)$ such that $\mathcal{Z}(b) \subset \mathcal{Z}(f - g^{2m}) = \mathcal{Z}(f)$ and $b^{2m}(f - g^{2m}) = \sum_{k \geq 1} a_k^2$; hence, $(bg)^{2m} + \sum_{k \geq 1} a_k^2 = b^{2m}f \in \mathfrak{a}$ and so $bg \in \sqrt[a]{\mathfrak{a}}$.

(iii) We can only assure in general that $\sqrt[a]{\mathfrak{a}} = \sqrt[b]{\mathfrak{a}}$ if $\sqrt[a]{\mathfrak{a}}$ is saturated and $\mathcal{Z}(\mathfrak{a})$ is an H^a -set.

5.1. Quasi-real ideals. Recall that we have seen that a convex ideal \mathfrak{a} verifies $\sqrt{\mathfrak{a}} = \sqrt[b]{\mathfrak{a}}$. The type of ideals which play a similar role with respect to the real radical has been classically defined as follows (see [AL, GT, BP]).

Definition and Lemma 5.5. Let (X, \mathcal{O}_X) be a real coherent reduced analytic space and let \mathfrak{a} be an ideal of $\mathcal{O}(X)$. We define the *square root* of \mathfrak{a} by

$$\sqrt[2]{\mathfrak{a}} := \left\{ f \in \mathcal{O}(X) : \exists a_i \in \mathcal{O}(X) \text{ such that } f^2 + \sum_{k \geq 1} a_k^2 \in \mathfrak{a} \right\}.$$

Then, $\mathfrak{a} \subset \sqrt[2]{\mathfrak{a}} \subset \sqrt[a]{\mathfrak{a}}$ and $\sqrt[a]{\mathfrak{a}} = \sqrt{\sqrt[2]{\mathfrak{a}}} = \bigcup_{k \geq 1} \sqrt[k]{\mathfrak{a}}$, where $\sqrt[k]{\mathfrak{a}} := \sqrt[2^{k-1}]{\sqrt{\mathfrak{a}}}$ for $k \geq 2$. Moreover, \mathfrak{a} is a real analytic ideal if and only if $\mathfrak{a} = \sqrt[2]{\mathfrak{a}}$.

Proof. The only non trivial point is to check that $\sqrt[2]{\mathfrak{a}}$ is closed under addition, which follows from the following classical trick that we recall here for the sake of completeness. Indeed, suppose that $f^2 + \sum_{k \geq 1} a_k^2, g^2 + \sum_{k \geq 1} b_k^2 \in \mathfrak{a}$. Thus,

$$(f+g)^2 + (f-g)^2 + 2\left(\sum_{k \geq 1} a_k^2 + \sum_{k \geq 1} b_k^2\right) = 2\left(f^2 + g^2 + \sum_{k \geq 1} a_k^2 + \sum_{k \geq 1} b_k^2\right) \in \mathfrak{a}$$

and so $f+g \in \sqrt[2]{\mathfrak{a}}$. \square

Next we explore the relations between the convex hull and the square root of an ideal \mathfrak{a} of $\mathcal{O}(X)$ whose zero-set $\mathcal{Z}(\mathfrak{a})$ is an $H^{\mathfrak{a}}$ -set (analogous statements hold when $\mathcal{Z}(\mathfrak{a})$ is an H -set). We define the ideal

$$\mathfrak{r}_2(\mathfrak{a}) := \{g \in \mathcal{O}(X) : \exists b \in \mathcal{O}(X) \text{ such that } \mathcal{Z}(b) \subset \mathcal{Z}(g) \text{ \& } bg \in \sqrt[2]{\mathfrak{a}}\}. \quad (5.7)$$

Remarks 5.6. Let \mathfrak{a} be an ideal of $\mathcal{O}(X)$ whose zero-set $\mathcal{Z}(\mathfrak{a})$ is an $H^{\mathfrak{a}}$ -set. Then,

- (i) In view of Theorem 3 and Lemma 5.5 we have $\mathcal{I}(\mathcal{Z}(\mathfrak{a})) = \widetilde{\sqrt[2]{\mathfrak{a}}} = \sqrt{\sqrt[2]{\mathfrak{a}}}$.
- (ii) Moreover, $(\sqrt[2]{\mathfrak{a}})^2 \subset \mathfrak{g}(\mathfrak{a}) \subset \mathfrak{r}_2(\mathfrak{a})$. The first inclusion is straightforward and for the second proceed similarly to the proof of Remark 5.4(ii).
- (iii) As a consequence of Remark 5.4(ii) we have $\sqrt{\mathfrak{g}(\mathfrak{a})} = \sqrt{\mathfrak{r}_2(\mathfrak{a})} = \sqrt[4]{\mathfrak{a}}$.

Lemma 5.7. *Let $X \subset \mathbb{R}^n$ be a global analytic set and let \mathfrak{a} be an ideal of $\mathcal{O}(X)$ whose zero set $\mathcal{Z}(\mathfrak{a})$ is an $H^{\mathfrak{a}}$ -set. Then,*

- (i) *If \mathfrak{a} is convex, then $\sqrt[2]{\mathfrak{a}}$ is also convex.*
- (ii) *If \mathfrak{a} is saturated, $(\sqrt[2]{\mathfrak{a}})^2 \subset (\mathfrak{g}(\mathfrak{a}))^2 \subset \mathfrak{g}(\mathfrak{a}) \subset \mathfrak{r}_2(\mathfrak{a})$.*

Proof. We begin by proving (i), let $g \in \mathcal{O}(X)$ and $f \in \sqrt[2]{\mathfrak{a}}$ such that $|g| \leq f$. Let $K \subset X$ be a compact set; by Lemma 2.3, there exists $h \in \mathcal{O}(X)$ such that $\mathcal{Z}(h) \cap K = \emptyset$ and $h^2 f \in \mathfrak{a}$. Since $|h^2 g| \leq h^2 f$ and \mathfrak{a} is convex, we deduce that $h^2 g \in \mathfrak{a}$; hence, by Lemma 2.3, we deduce that $g \in \sqrt[2]{\mathfrak{a}}$ and so $\sqrt[2]{\mathfrak{a}}$ is convex.

Next, observe that (ii) follows straightforwardly from Lemma 4.8 and Remarks 5.6. \square

We can unify the two notions of convex hull $\mathfrak{g}(\mathfrak{a})$ and square root $\sqrt[2]{\mathfrak{a}}$ of an ideal \mathfrak{a} of $\mathcal{O}(X)$ under the following general concept. In [GT] appears a similar definition concerning the *defining ideals*.

Definition 5.8. Let (X, \mathcal{O}_X) be a real coherent reduced analytic space. We say that an ideal \mathfrak{a} of $\mathcal{O}(X)$ is *quasi-real* if its radical $\sqrt{\mathfrak{a}}$ is a real analytic ideal.

Remarks 5.9. (i) Let \mathfrak{a} be a quasi-real saturated ideal and let $\mathfrak{a} = \bigcap_{i \in I} \mathfrak{q}_i$ be a normal primary decomposition of \mathfrak{a} . Let $i_0 \in I$ be such that $\sqrt{\mathfrak{q}_{i_0}}$ is a minimal ideal of the family $\{\sqrt{\mathfrak{q}_i}\}_{i \in I}$. Then, \mathfrak{q}_{i_0} is a quasi-real saturated prime ideal.

Proof. Indeed, let $h_{i_0} \in \bigcap_{j \neq i_0} \mathfrak{q}_j \setminus \sqrt{\mathfrak{q}_{i_0}}$ (see the proof of Remark 4.7(i)). We have to prove that $\sqrt{\mathfrak{q}_{i_0}}$ is a real analytic ideal. Let $a_k \in \mathcal{O}(\mathbb{R}^n)$ be such that $f = \sum_{k \geq 1} a_k \in \sqrt{\mathfrak{p}_{i_0}}$. Then, there exists $m \geq 1$ such that $f^m \in \mathfrak{p}_{i_0}$. Thus, for each k there exists a sum of squares σ_k in $\mathcal{O}(\mathbb{R}^m)$ such that $a_k^{2m} + \sigma_k \in \mathfrak{q}_{i_0}$; hence, $h_{i_0}^2 a_k^{2m} + h_{i_0}^2 \sigma_k \in \mathfrak{a}$. Therefore, since \mathfrak{a} is quasi-radical, $h_{i_0} a_k^m \in \mathfrak{q}_{i_0}$. But since $h_{i_0} \notin \sqrt{\mathfrak{q}_{i_0}}$, we deduce that there exist $\ell \geq 1$ such that $a_k^{m\ell} \in \mathfrak{q}_{i_0}$ and so $a_k \in \sqrt{\mathfrak{q}_{i_0}}$. This means that $\sqrt{\mathfrak{q}_{i_0}}$ is a real ideal and so \mathfrak{q}_{i_0} is quasi-radical. \square

(ii) Under the hypotheses of Remark 4.7(iii), the result is no longer true in general if $\sqrt{\mathfrak{q}_{i_0}}$ is not a minimal ideal of the family $\{\sqrt{\mathfrak{q}_i}\}_{i \in I}$. For an example use the same already studied in Remark 4.7(iv).

(iii) Suppose now that \mathfrak{a} is moreover quasi-real and saturated and let $\mathfrak{a} = \bigcap_{i \in I} \mathfrak{q}_i$ be a normal primary decomposition of \mathfrak{a} . Let J be the set of indices $j \in I$ such that $\sqrt{\mathfrak{q}_j}$ is a minimal element of the family $\{\sqrt{\mathfrak{q}_i}\}_{i \in I}$; by Remark 5.9(i), each primary ideal \mathfrak{q}_j is quasi-real. It holds that

$$\widetilde{\sqrt{\mathfrak{a}}} = \widetilde{\sqrt{\mathfrak{a}}^{\text{ar}}} = \mathcal{I}(\mathcal{Z}(\mathfrak{a})) = \bigcap_{j \in J} \sqrt{\mathfrak{q}_j} = \bigcap_{i \in I} \sqrt{\mathfrak{q}_i}.$$

(iv) If we are working in the framework of convex saturated ideals it holds an analogous result to Theorem 2.2 just changing “Stein space” by “global analytic set” and “closed ideal” by “quasi-real saturated ideal of $\mathcal{O}(X)$ whose zero-set is either an H -set or an $H^{\mathfrak{a}}$ -set”. As one can expect, the proof runs analogously to the one of Theorem 2.2 (see [F, §5.Satz 9]) and we left the concrete details to the reader. For further comments revisit Remarks 5.4.

6. REAL NULLSTELLENSÄTZE AND COMPLEX ANALYTIC GERMS AT \mathbb{R}^n

6.1. Saturated primary ideals and complex analytic germs at \mathbb{R}^n . Let $X \subset \mathbb{R}^n$ be a global analytic set. As we have already commented $\mathcal{O}(X) = \mathcal{O}(\mathbb{R}^n)/\mathcal{I}(X)$; hence, by means of the correspondence theorem for ideals, every ideal \mathfrak{a} of $\mathcal{O}(X)$ is the quotient by $\mathcal{I}(X)$ of an ideal $\mathfrak{b} \subset \mathcal{O}(\mathbb{R}^n)$ which contains $\mathcal{I}(X)$. Thus, for the purposes of this section, we assume that $X = \mathbb{R}^n$.

Definition 6.1. Let $\mathfrak{a} \subset \mathcal{O}(\mathbb{R}^n)$ be a saturated ideal. We extend the coherent sheaf $\mathfrak{a}\mathcal{O}_X$ to a coherent sheaf of ideals \mathcal{F} on invariant open Stein neighborhood Ω of \mathbb{R}^n in \mathbb{C}^n . The analytic germ $Y_{\mathbb{R}^n}$ at \mathbb{R}^n of the support $Y := \text{supp}(\mathcal{F})$ will be called the *complex zero-set* $\mathcal{Z}_{\mathbb{C}}(\mathfrak{a})$ of \mathfrak{a} .

Lemma 6.2. Let $\mathfrak{q} \subset \mathcal{O}(\mathbb{R}^n)$ be a primary saturated ideal. Then, $f \in \mathfrak{p} := \sqrt{\mathfrak{q}}$ if and only if there exists an open neighbourhood Ω of \mathbb{R}^n in \mathbb{C}^n , a holomorphic extension F of f to Ω and a representant Y of $\mathcal{Z}_{\mathbb{C}}(\mathfrak{q})$ in Ω such that $Y \subset \mathcal{Z}(F)$; in other words, $f \in \mathfrak{p} := \sqrt{\mathfrak{q}}$ if and only if F vanishes on $\mathcal{Z}_{\mathbb{C}}(\mathfrak{q})$.

Proof. The “only if” implication follows from the fact that \mathfrak{q} is saturated; hence, \mathfrak{p} is saturated also, and $f \in \mathcal{I}(\mathcal{Z}(\mathfrak{q}))$ implies F vanishes on $\mathcal{Z}_{\mathbb{C}}(\mathfrak{q})$. For the “if” implication, let Y be a representant of $\mathcal{Z}_{\mathbb{C}}(\mathfrak{q})$ on a suitable complex neighbourhood such that $Y \subset \mathcal{Z}(F)$. Pick a point

$$x \in \mathcal{Z}(\mathfrak{q}) = \mathcal{Z}_{\mathbb{C}}(\mathfrak{q}) \cap \mathbb{R}^n = Y \cap \mathbb{R}^n \subset \mathcal{Z}(F) \cap \mathbb{R}^n = \mathcal{Z}(f).$$

Since $Y \subset \mathcal{Z}(F)$, we have $F_x \in \mathcal{I}(\mathcal{Z}(Y_x)) = \mathcal{I}(\mathcal{Z}(\mathfrak{q}_x \mathcal{O}_{\mathbb{C}^n, x})) = \sqrt{\mathfrak{q}_x \mathcal{O}_{\mathbb{C}^n, x}}$; hence, there exists $m \geq 1$ such that $F_x^m \in \mathfrak{q}_x \mathcal{O}_{\mathbb{C}^n, x}$. Thus, by Lemma 1.1, $f^m = (F|_{\mathbb{R}^n})^m \in \mathfrak{q}$ and so $f \in \mathfrak{p}$. \square

Remarks 6.3. (i) Let $\mathfrak{a}_1, \mathfrak{a}_2$ be two saturated ideals of $\mathcal{O}(X)$ such that $\mathfrak{a}_1 \subset \mathfrak{a}_2$. Then, $\mathcal{Z}_{\mathbb{C}}(\mathfrak{a}_2) \subset \mathcal{Z}_{\mathbb{C}}(\mathfrak{a}_1)$.

(ii) Let $\mathfrak{q}_1, \mathfrak{q}_2$ be two saturated primary ideals of $\mathcal{O}(X)$ such that $\mathcal{Z}_{\mathbb{C}}(\mathfrak{q}_2) \subset \mathcal{Z}_{\mathbb{C}}(\mathfrak{q}_1)$. Then, as a straightforward consequence, we have $\mathfrak{p}_1 := \sqrt{\mathfrak{q}_1} \subset \sqrt{\mathfrak{q}_2} =: \mathfrak{p}_2$.

Lemma 6.4. *Let $\mathfrak{q} \subset \mathcal{O}(\mathbb{R}^n)$ be a primary saturated ideal. Then, there exist an irreducible analytic germ $Z_{\mathbb{R}^n}$ such that $\mathcal{Z}_{\mathbb{C}}(\mathfrak{q}) = Z_{\mathbb{R}^n} \cup \sigma(Z_{\mathbb{R}^n})$. In particular, if $\mathcal{Z}_{\mathbb{C}}(\mathfrak{q})$ is invariant, it is also irreducible.*

Proof. We extend the coherent sheaf $\mathfrak{q}\mathcal{O}_X$ to a coherent sheaf of ideals \mathcal{F} on an invariant open Stein neighborhood Ω of \mathbb{R}^n in \mathbb{C}^n and denote $Y := \text{supp}(\mathcal{F})$; recall that $\mathcal{Z}_{\mathbb{C}}(\mathfrak{q}) = Y_{\mathbb{R}^n}$. We may assume that Ω is moreover contractible. Consider the subring $\mathcal{A}(\Omega)$ of $H^0(\Omega, \mathcal{O}_{\mathbb{C}^n})$ of all invariant holomorphic functions on Ω . Observe that the restriction homomorphism $\varphi : \mathcal{A}(\Omega) \rightarrow \mathcal{O}(\mathbb{R}^n)$, $F \mapsto F|_{\mathbb{R}^n}$ is injective. Since \mathfrak{q} is a primary ideal, $\mathfrak{p} := \sqrt{\mathfrak{q}}$ is a prime ideal and so $\mathfrak{P} := \varphi^{-1}(\mathfrak{p})$ is also prime.

Observe that since $\mathcal{Z}(\mathfrak{p}) = \mathcal{Z}(\mathfrak{q}) \neq \emptyset$, then $\mathcal{Z}(\mathfrak{P}) \neq \emptyset$. Now, by Cartan's Theorem A and using that \mathfrak{q} is saturated, we deduce, maybe after shrinking Ω , that $Y_{\mathbb{R}^n} = \mathcal{Z}(\mathfrak{P})_{\mathbb{R}^n}$ and $Y = \mathcal{Z}(\mathfrak{P})$.

Next, let $Y_{\mathbb{R}^n} = \bigcup_{i \in I} Z_{i, \mathbb{R}^n}$ be the decomposition of $Y_{\mathbb{R}^n}$ as the union of its irreducible components. Pick one of them, and for simplicity denote it by $Z_{\mathbb{R}^n}$; by [WB, Cor.2, pag.151] (and its proof), we may assume that there exists an irreducible analytic set Z in Ω whose germ at \mathbb{R}^n is precisely $Z_{\mathbb{R}^n}$. Notice that Z and $\sigma(Z)$ are (maybe equal) irreducible components of Y because $Z_{\mathbb{R}^n}$ is an irreducible component of the invariant germ $Y_{\mathbb{R}^n}$. Assume that $Y \neq Z \cup \sigma(Z)$ and let T be the union of all the other irreducible components of Y ; clearly, T is invariant. Choose now invariant $F, G \in H^0(\Omega, \mathcal{O}_{\mathbb{C}^n})$ such that:

- $Z \cup \sigma(Z) \subset \mathcal{Z}(F)$ but $T \not\subset \mathcal{Z}(F)$,
- $T \subset \mathcal{Z}(G)$ but $Z \cup \sigma(Z) \not\subset \mathcal{Z}(G)$.

Thus, the invariant holomorphic function FG vanishes on Y .

Let $x \in \mathcal{Z}(\mathfrak{p}) = Y \cap \mathbb{R}^n$ and observe that by the complex local analytic Nullstellensatz

$$\mathcal{I}(Y_x) = \mathcal{I}(\mathcal{Z}(\mathcal{F}_x)) = \mathcal{I}(\mathcal{Z}(\mathfrak{q}_x \mathcal{O}_{\mathbb{C}^n, x})) = \sqrt{\mathfrak{q}_x \mathcal{O}_{\mathbb{C}^n, x}}.$$

Thus, there exists $m \geq 1$ such that $(FG)_x^m \in \mathfrak{q}_x \mathcal{O}_{\mathbb{C}^n, x}$; hence, by 1.1, $(FG)^m \in \mathfrak{q}$ and so $FG \in \mathfrak{p} \cap \mathcal{A}(\Omega) = \mathfrak{P}$. Thus, since this last ideal is prime, we deduce that either $F \in \mathfrak{P}$ or $G \in \mathfrak{P}$; hence, $T \subset Y = \mathcal{Z}(\mathfrak{P}) \subset \mathcal{Z}(F)$, a contradiction. In this way, we conclude that $Y = Z \cup \sigma(Z)$ and so $Y_{\mathbb{R}^n} = Z_{\mathbb{R}^n} \cup \sigma(Z_{\mathbb{R}^n})$, as wanted. \square

Lemma 6.5. *Let $\mathfrak{a} \subset \mathcal{O}(\mathbb{R}^n)$ be a saturated ideal and let $\mathfrak{a} = \bigcap_{i \in I} \mathfrak{q}_i$ be a normal primary decomposition of \mathfrak{a} and let $J \subset I$ be the collection of the indices corresponding to the isolated primary components of \mathfrak{a} . Then, $\mathcal{Z}_{\mathbb{C}}(\mathfrak{a}) = \bigcup_{j \in J} \mathcal{Z}_{\mathbb{C}}(\mathfrak{q}_j)$ and for each $j \in J$ there exists an irreducible component Z_{j, \mathbb{R}^n} of $\mathcal{Z}_{\mathbb{C}}(\mathfrak{a})$ such that $\mathcal{Z}_{\mathbb{C}}(\mathfrak{q}_j) = Z_{j, \mathbb{R}^n} \cup \sigma(Z_{j, \mathbb{R}^n})$.*

Proof. Observe first that

$$\mathcal{Z}_{\mathbb{C}}(\mathfrak{a}) \cap \mathbb{R}^n = \mathcal{Z}(\mathfrak{a}) = \bigcup_{i \in I} \mathcal{Z}(\mathfrak{q}_i) = \bigcup_{j \in J} \mathcal{Z}(\mathfrak{q}_j) = \bigcup_{j \in J} \mathcal{Z}_{\mathbb{C}}(\mathfrak{q}_j) \cap \mathbb{R}^n.$$

Now, fix $x \in Z \cap \mathbb{R}^n$ and observe that $\mathcal{Z}_{\mathbb{C}}(\mathfrak{a})_x = \mathcal{Z}(\mathfrak{a} \mathcal{O}_{\mathbb{C}^n, x})$. Let $\mathfrak{q}_{i_1}, \dots, \mathfrak{q}_{i_r}$ be the primary ideals of our normal primary decomposition whose zero-sets contain x ; we may assume that $\mathfrak{q}_{i_1}, \dots, \mathfrak{q}_{i_s}$ are those primary ideals among $\mathfrak{q}_{i_1}, \dots, \mathfrak{q}_{i_r}$ that are moreover isolated.

Observe that

$$\begin{aligned}\mathcal{Z}_{\mathbb{C}}(\mathfrak{a})_x &= \mathcal{Z}(\mathfrak{a}\mathcal{O}_{\mathbb{C}^n, x}) = \mathcal{Z}\left(\bigcap_{\ell=1}^r \mathfrak{q}_{i_\ell} \mathcal{O}_{\mathbb{C}^n, x}\right) = \bigcup_{\ell=1}^r \mathcal{Z}(\sqrt{\mathfrak{q}_{i_\ell}} \mathcal{O}_{\mathbb{C}^n, x}) \\ &= \bigcup_{\ell=1}^s \mathcal{Z}(\sqrt{\mathfrak{q}_{i_\ell}} \mathcal{O}_{\mathbb{C}^n, x}) = \bigcup_{\ell=1}^s \mathcal{Z}(\mathfrak{q}_{i_\ell} \mathcal{O}_{\mathbb{C}^n, x}) = \bigcup_{\ell=1}^s \mathcal{Z}_{\mathbb{C}}(\mathfrak{q}_{i_\ell})_x = \bigcup_{j \in J} \mathcal{Z}_{\mathbb{C}}(\mathfrak{q}_j)_x;\end{aligned}$$

hence, $\mathcal{Z}_{\mathbb{C}}(\mathfrak{a}) = \bigcup_{j \in J} \mathcal{Z}_{\mathbb{C}}(\mathfrak{q}_j)$.

By Lemma 6.4, we know that for each $\mathcal{Z}_{\mathbb{C}}(\mathfrak{q}_j)$, there exist an irreducible analytic germ Z_{j, \mathbb{R}^n} at \mathbb{R}^n such that $\mathcal{Z}_{\mathbb{C}}(\mathfrak{q}_j) = Z_{j, \mathbb{R}^n} \cup \sigma(Z_{j, \mathbb{R}^n})$; hence, $\mathcal{Z}_{\mathbb{C}}(\mathfrak{a}) = \bigcup_{j \in J} Z_{j, \mathbb{R}^n} \cup \sigma(Z_{j, \mathbb{R}^n})$. By Remarks 6.3 and the fact that the primary ideals \mathfrak{q}_j are isolated, we deduce that $Z_{j, \mathbb{R}^n} \not\subset Z_{j', \mathbb{R}^n} \cup \sigma(Z_{j', \mathbb{R}^n})$ if $j \neq j'$ and so for each $j \in J$, the germs Z_{j, \mathbb{R}^n} and $\sigma(Z_{j, \mathbb{R}^n})$ are irreducible components of $\mathcal{Z}_{\mathbb{C}}(\mathfrak{a})$. We are done. \square

Now, we are ready to prove Theorem 4.

Proof of Theorem 4. First, by Lemma 6.4, there exists an irreducible analytic germ $Z_{\mathbb{R}^n}$ such that $\mathcal{Z}_{\mathbb{C}}(\mathfrak{q}) = Z_{\mathbb{R}^n} \cup \sigma(Z_{\mathbb{R}^n})$. Now, we prove the following implications.

(i) \implies (ii) Since $\mathcal{I}(\mathcal{Z}(\mathfrak{q})) = \sqrt{\mathfrak{q}}$, we deduce by [WB, pag.154] that $\mathcal{Z}_{\mathbb{C}}(\sqrt{\mathfrak{q}}) = \mathcal{Z}_{\mathbb{C}}(\mathfrak{q})$ is the germ at \mathbb{R}^n of the “complexification” of $\mathcal{Z}(\sqrt{\mathfrak{q}}) = \mathcal{Z}(\mathfrak{q})$. Now, since the dimension of the “complexification” of $\mathcal{Z}(\mathfrak{q})$ coincides with its dimension (see [WB, §8. Prop.12]), we deduce that $\dim \mathcal{Z}_{\mathbb{C}}(\mathfrak{q}) = \dim \mathcal{Z}(\mathfrak{q})$.

(ii) \implies (i) Let $Y_{\mathbb{R}^n}$ be the germ at \mathbb{R}^n of the “complexification” of $\mathcal{Z}(\mathfrak{q})$. By [WB, pag.154], we have $Y_{\mathbb{R}^n} \subset Z_{\mathbb{R}^n} \cup \sigma(Z_{\mathbb{R}^n})$. Since $Z_{\mathbb{R}^n}$ is irreducible, we have that either $Z_{\mathbb{R}^n} = \sigma(Z_{\mathbb{R}^n})$ or $\dim(Z_{\mathbb{R}^n} \cap \sigma(Z_{\mathbb{R}^n})) < \dim Z_{\mathbb{R}^n}$. But this last is imposible because if that is the case

$$\begin{aligned}\dim \mathcal{Z}(\mathfrak{q}) &= \dim Y_{\mathbb{R}^n} \leq \dim(Z_{\mathbb{R}^n} \cap \sigma(Z_{\mathbb{R}^n})) < \dim Z_{\mathbb{R}^n} \\ &\leq \dim(Z_{\mathbb{R}^n} \cup \sigma(Z_{\mathbb{R}^n})) = \dim \mathcal{Z}_{\mathbb{C}}(\mathfrak{q}) = \dim \mathcal{Z}(\mathfrak{q}),\end{aligned}$$

a contradiction. Thus, $\mathcal{Z}_{\mathbb{C}}(\mathfrak{q}) = Z_{\mathbb{R}^n}$ and

$$\dim \mathcal{Z}(\mathfrak{q}) = \dim Y_{\mathbb{R}^n} \leq \dim Z_{\mathbb{R}^n} = \dim \mathcal{Z}_{\mathbb{C}}(\mathfrak{q}) = \dim \mathcal{Z}(\mathfrak{q});$$

hence, $\dim Y_{\mathbb{R}^n} = \dim Z_{\mathbb{R}^n}$ and $Z_{\mathbb{R}^n}$ being irreducible, we conclude that $Y_{\mathbb{R}^n} = Z_{\mathbb{R}^n}$. Thus, by Lemma 6.2, $f \in \sqrt{\mathfrak{q}}$ if and only if there exists an open neighbourhood Ω of \mathbb{R}^n in \mathbb{C}^n , a holomorphic extension F of f to Ω and a complex analytic subset $T \subset \mathcal{Z}(F)$ in Ω such that $T_{\mathbb{R}^n} = \mathcal{Z}_{\mathbb{C}}(\mathfrak{q}) = Z_{\mathbb{R}^n}$.

On the other hand, by [WB, pag.154], we have that $g \in \mathcal{I}(\mathcal{Z}(\mathfrak{q}))$ if and only if there exists an open neighbourhood Ω of \mathbb{R}^n in \mathbb{C}^n , a holomorphic extension G of g to Ω and a complex analytic subset $S \subset \mathcal{Z}(G)$ in Ω such that $S_{\mathbb{R}^n} = Y_{\mathbb{R}^n}$.

Thus, since $Z_{\mathbb{R}^n} = Y_{\mathbb{R}^n}$, we conclude that $\mathcal{I}(\mathcal{Z}(\mathfrak{q})) = \sqrt{\mathfrak{q}}$.

(ii) \implies (iii) It is straightforward.

(iii) \implies (ii) Let (Ω, Z) be such that Ω is an open invariant neighborhood of \mathbb{R}^n in \mathbb{C}^n and Z is an irreducible representative of $Z_{\mathbb{R}^n}$ in Ω (see [WB, Cor.2, pag.151]). The

irreducibility of Z guarantees that it is pure dimensional; hence, so is $Z \cup \sigma(Z)$. We have

$$\begin{aligned} \dim \mathcal{Z}(\mathfrak{q}) &\geq \dim \mathcal{Z}(\mathfrak{q})_x = \dim \mathcal{Z}(\mathfrak{q}\mathcal{O}_{\mathbb{R}^n, x}) = \dim \mathcal{Z}(\mathfrak{q}\mathcal{O}_{\mathbb{C}^n, x}) \\ &= \dim(Z_x \cup \sigma(Z)_x) = \dim(Z_{\mathbb{R}^n} \cup \sigma(Z_{\mathbb{R}^n})) = \dim(\mathcal{Z}_{\mathbb{C}}(\mathfrak{q})) \geq \dim \mathcal{Z}(\mathfrak{q}), \end{aligned}$$

and so, $\dim \mathcal{Z}_{\mathbb{C}}(\mathfrak{q}) = \dim \mathcal{Z}(\mathfrak{q})$. \square

Next, we introduce the following invariant concerning the ideals of $\mathcal{O}(\mathbb{R}^n)$.

Definition 6.6. We say that a finite set $\mathfrak{F} := \{f_1, \dots, f_m\} \subset \mathcal{O}(\mathbb{R}^n)$ is *sharp* if the dimension $\dim \mathcal{Z}_{\mathbb{C}}(f_1, \dots, f_m) = n - m$.

Remarks 6.7. (i) Let $\mathfrak{F} := \{f_1, \dots, f_m\} \subset \mathcal{O}(\mathbb{R}^n)$. For each $\ell = 1, \dots, m$ the (finitely generated) ideal $\mathfrak{b}_\ell = (f_1, \dots, f_\ell)\mathcal{O}(X)$ is saturated and so admits a normal primary decomposition $\mathfrak{b}_\ell = \bigcap_{j \in J_\ell} \mathfrak{q}_{j\ell}$. Then, \mathfrak{F} is a sharp family if and only if f_ℓ does not belong to any of the minimal prime ideals of the family $\{\sqrt{\mathfrak{q}_{j, \ell-1}}\}_{j \in J_{\ell-1}}$ for each $\ell = 2, \dots, m$.

Let Ω be an open neighbourhood of \mathbb{R}^n in \mathbb{C} on which each f_i admits a holomorphic extension F_i . Recall the following well-known consequence of the Identity Principle:

(6.7.1) *If Y is an irreducible complex analytic subset of Ω , then Y is pure dimensional and if $F \in H^0(\Omega, \mathcal{O}_{\mathbb{C}^n})$, then either $Y \subset \mathcal{Z}(F)$ or $\dim(Y \cap \mathcal{Z}(F)) < \dim Y$.*

Thus, shrinking the open set Ω in each step if necessary, it follows from 6.7.1 that $\dim \mathcal{Z}_{\mathbb{C}}(f_1, \dots, f_m) \geq n - m$. A careful analysis shows, by means of Lemmata 6.4 and 6.5 and 6.7.1, that $\dim \mathcal{Z}_{\mathbb{C}}(f_1, \dots, f_m) = n - m$ if and only if f_ℓ does not belong to any of the minimal prime ideals of the family $\{\sqrt{\mathfrak{q}_{j, \ell-1}}\}_{j \in J_{\ell-1}}$ for each $\ell = 2, \dots, m$. Since this kind of argument is standard, we left the concrete details to the reader.

(ii) If $\mathfrak{a} \subset \mathcal{O}(\mathbb{R}^n)$ is an ideal, we have

$$\sup\{\text{card}(\mathfrak{F}) : \mathfrak{F} \subset \mathfrak{a} \text{ is sharp}\} = n - \dim \mathcal{Z}_{\mathbb{C}}(\mathfrak{a}) \leq n - \dim \mathcal{Z}(\mathfrak{a})$$

and if \mathfrak{q} is a primary ideal, we have (by Theorem 4) that

$$\mathcal{I}(\mathcal{Z}(\mathfrak{q})) = \sqrt{\mathfrak{q}} \iff \sup\{\text{card}(\mathfrak{F}) : \mathfrak{F} \subset \mathfrak{q} \text{ is sharp}\} = n - \dim \mathcal{Z}(\mathfrak{q}).$$

(iii) If \mathfrak{q} verifies the equivalent conditions of Theorem 4, then

$$\dim \mathcal{Z}(\mathfrak{q}) = \sup\{\text{card}(\mathfrak{F}) : \mathfrak{F} \subset \mathfrak{q} \text{ is sharp}\}.$$

Remark 6.8. Let $\mathfrak{q} \subset \mathcal{O}(\mathbb{R}^n)$ be a primary saturated ideal. Then, \mathfrak{q} is a principal ideal if and only if $\sqrt{\mathfrak{q}}$ is a principal ideal.

For the “if” implication, assume that $\sqrt{\mathfrak{q}}$ is a principal ideal generated by $f \in \mathcal{O}(\mathbb{R}^n)$; once can check that \mathfrak{q} is generated by f^k where $k = \min\{m \geq 1 : f^m \in \mathfrak{q}\}$.

Conversely, assume that \mathfrak{q} is generated by $f \in \mathcal{O}(\mathbb{R}^n)$. By [Ca, Prop.3], there exists $h \in \mathcal{O}(\mathbb{R}^n)$ such that $h_x \mathcal{O}_x = \sqrt{f_x \mathcal{O}_x}$ for each point $x \in \mathbb{R}^n$. We claim that $\sqrt{\mathfrak{q}} = h\mathcal{O}(\mathbb{R}^n)$.

Indeed, if $g \in \sqrt{\mathfrak{q}} = \sqrt{f\mathcal{O}(\mathbb{R}^n)}$, then the germ g_x is in $\sqrt{f_x \mathcal{O}_x} = h_x \mathcal{O}_x$ for each $x \in \mathbb{R}^n$ and so $g \in (h)$. Now, we prove that $h \in \sqrt{\mathfrak{q}}$. Pick a point $x \in \mathcal{Z}(\mathfrak{q})$; since $h_x \mathcal{O}_x = \sqrt{f_x \mathcal{O}_x}$, we find an integer m such that $h_x^m \in f_x \mathcal{O}_x = \mathfrak{q} \mathcal{O}_x$. Being \mathfrak{q} a saturated primary ideal, Lemma 1.1 ensures that $h^m \in \mathfrak{q}$, as we wanted.

Corollary 6.9. *Let $\mathfrak{q} \subset \mathcal{O}(\mathbb{R}^n)$ be a primary saturated ideal. We have*

- (i) *If $\dim(\mathcal{Z}(\mathfrak{q})) = n - 1$, then $\mathcal{I}(\mathcal{Z}(\mathfrak{q})) = \sqrt{\mathfrak{q}}$.*
- (ii) *If $\dim(\mathcal{Z}(\mathfrak{q})) = n - 2$, then $\mathcal{I}(\mathcal{Z}(\mathfrak{q})) = \sqrt{\mathfrak{q}}$ if and only if \mathfrak{q} is not principal.*

6.2. Special factors. To finish we would like to insist in the fact that the obstruction to solve Hilbert's 17th Problem and to state the real Nullstellensatz in terms of the real analytic radical is concentrated in the special factors. To strong this fact we observe the following.

Lemma 6.10. *Let $f \in \mathcal{O}(\mathbb{R}^n)$ be an analytic function which is a (possibly infinite) sum of squares of meromorphic functions on \mathbb{R}^n . Then, the ideal $\mathfrak{a} := f\mathcal{O}(\mathbb{R}^n)$ is not real analytic.*

Proof. By [ABFR3, 4.1] there exist $h_0, h_k \in \mathcal{O}(\mathbb{R}^n)$ such that $\mathcal{Z}(h_0) \subset \mathcal{Z}(f)$ and $h_0^2 f = \sum_{k \geq 1} h_k^2$. Let $m \geq 0$ be the maximum integer such that f^m divides each h_k for $k \geq 1$. We write $h_0^2 f = f^{2m} \sum_{k \geq 1} h'_k{}^2$ for some $h'_k \in \mathcal{O}(\mathbb{R}^n)$; hence, f^m divides h_0 and we have $h_0^2 f^{2m+1} = f^{2m} \sum_{k \geq 1} h'_k{}^2$ for some $h'_0 \in \mathcal{O}(\mathbb{R}^n)$. Simplifying, $h_0^2 f = \sum_{k \geq 1} h'_k{}^2$. Assume, by way of contradiction, that \mathfrak{a} is real analytic, then f divides h'_k for all $k \geq 1$, a contradiction. \square

Remarks 6.11. (i) Let $f \in \mathcal{O}(\mathbb{R}^n)$ be an analytic function such that $\dim \mathcal{Z}(f) \leq n - 2$ and the principal saturated ideal $f\mathcal{O}(\mathbb{R}^n)$ is primary. Then, there exist an irreducible analytic function $g \in \mathcal{O}(\mathbb{R}^n)$ and an integer $m \geq 1$ such that $f = \pm g^m$.

Indeed, let $h \in \mathcal{O}(\mathbb{R}^n)$ be such that $\sqrt{f\mathcal{O}(\mathbb{R}^n)} = h\mathcal{O}(\mathbb{R}^n)$ (see Remark 6.8); since h generates a prime ideal, it is irreducible. Notice that $f\mathcal{O}(\mathbb{R}^n) = h^m\mathcal{O}(\mathbb{R}^n)$ for $m = \min\{k \geq 1 : h^k \in f\mathcal{O}(\mathbb{R}^n)\}$; hence, f and h^m coincide up to a unit and the statement follows.

(ii) On the other hand, it seems that a modification of the arguments developed in [De] shows that given a connected global analytic set $Z \subset \mathbb{R}^n$ of codimension ≥ 2 , there exists a special factor whose zero-set is Z . Thus, it seems that any advance on the representation as sums of squares of the special factors of $\mathcal{O}(\mathbb{R}^n)$ will provide light to understand the still open challenging problems in Real Analytic Geometry.

REFERENCES

- [ABF] F. Acquistapace, F. Broglia, J.F. Fernando: On Hilbert's 17th Problem and Pfister's multiplicative formulae for the ring of real analytic functions. *Ann. Sc. Norm. Super. Pisa Cl. Sci.* (5) **XXX** (2012, accepted), no. X, XXX-XXX.
- [ABFR1] F. Acquistapace, F. Broglia, J.F. Fernando, J.M. Ruiz: On the Pythagoras numbers of real analytic surfaces, *Ann. Sci. École Norm. Sup.* **38** (2005), no. 5, 751-772.
- [ABFR2] F. Acquistapace, F. Broglia, J.F. Fernando, J.M. Ruiz: On the Pythagoras numbers of real analytic curves. *Math. Z.* **257** (2007) no. 1, 13-21.
- [ABFR3] F. Acquistapace, F. Broglia, J.F. Fernando, J.M. Ruiz: On the finiteness of Pythagoras numbers of real meromorphic functions, *Bull. Soc. Math. France* **138** (2010), no. 2, 291-307.
- [ABS] F. Acquistapace, F. Broglia, M. Shiota: The finiteness property and Łojasiewicz inequality for global semianalytic sets. *Adv. in Geom.* **5** (2005), 453-466.
- [A] W.A. Adkins: A real analytic Nullstellensatz for two dimensional manifolds. *Boll. Un. Mat. Ital.* **14B** (1977) no. 5, 888-903.
- [AL] W.A. Adkins, J.V. Leahy: A global real analytic Nullstellensatz. *Duke Math. Journal* **43** (1976), no. 1, 81-86.
- [dB1] P. de Bartolomeis: Algebre di Stein nel caso reale. *Rend. Accad. Naz.* **XL** no. 5 1/2 (1975/76), 105-144 (1977).
- [dB2] P. de Bartolomeis: Una nota sulla topologia delle algebre reali coerenti. *Boll. Un. Mat. Ital.* **13A** (1976) no. 5, 123-125.
- [BM] E. Bierstone, P.D. Milman: Semianalytic and subanalytic sets. *Inst. Hautes Études Sci. Publ. Math.* **67** (1988), 5-42.

- [BCR] J. Bochnak, M. Coste, M.F. Roy: Real algebraic geometry. Translated from the 1987 French original. Revised by the authors. *Ergebnisse der Mathematik und ihrer Grenzgebiete* (3), **36**. Springer-Verlag, Berlin, 1998.
- [BKS] J. Bochnak, W.Kucharz, M.Shiota: On equivalence of ideals of real global analytic functions and the 17th Hilbert problem. *Invent. Math.* **63** (1981), no. 3, 403–421.
- [BP] F. Broglia, F. Pieroni: The Nullstellensatz for real coherent analytic surfaces. *Rev. Mat. Iberoam.* **25** (2009), no. 2, 781–798.
- [C1] H. Cartan: Idéaux et modules de fonctions analytiques de variables complexes. *Bull. Soc. Math. France* **78**, (1950), 29–64.
- [C2] H. Cartan: Variétés analytiques réelles et variétés analytiques complexes. *Bull. Soc. Math. France* **85** (1957), 77–99.
- [Ca] B.E. Cain: A two color theorem for analytic maps in \mathbb{R}^n . *Proc. Amer. Math. Soc.* **39** (1973), 261–266.
- [D] J.P. D’Angelo: Real and complex geometry meet the Cauchy-Riemann equations. *Analytic and algebraic geometry*, 77–182, IAS/Park City Math. Ser., **17**, Amer. Math. Soc., Providence, RI, 2010.
- [DM] C.N. Delzell, J.J. Madden: Lattice-ordered rings and semialgebraic geometry. I. *Real analytic and algebraic geometry* (Trento, 1992), 103–129, de Gruyter, Berlin, 1995.
- [De] J.P. Demailly: Construction d’hypersurfaces irréductibles avec lieu singulier donné dans \mathbb{C}^n . *Ann. Inst. Fourier (Grenoble)* **30** (1980), no. 3, 219–236.
- [Fe] J.F. Fernando: On the Hilbert’s 17th problem for global analytic functions on dimension 3, *Comment. Math. Helv.* **83** (2008), no. 1, 67–100.
- [FG] J.F. Fernando, J.M. Gamboa: Real Algebra from Hilbert’s 17th Problem. Dip. Mat. Univ. Pisa, *Dottorato di Ricerca in Matematica*, Istituti Editoriali e Poligrafici Internazionali, Pisa (2012, to appear).
- [F] O. Forster: Primärzerlegung in Steinschen Algebren. *Math. Ann.* **154** (1964), 307–329.
- [GT] M. Galbiati, A. Tognoli: Alcune proprietà delle varietà algebriche reali. *Ann. Scuola Norm. Sup. Pisa* (3) **27** (1973), 359–404 (1974).
- [GR] R. Gunning, H. Rossi: Analytic functions of several complex variables. Englewood Cliff: Prentice Hall, 1965.
- [H] M.W. Hirsch: Differential topology. *Graduate Texts in Mathematics*, **33**. Springer-Verlag, New York–Heidelberg–Berlin: 1976.
- [Jw] P. Jaworski: Extensions of orderings on fields of quotients of rings of real analytic functions. *Math. Nachr.* **125** (1986), 329–339.
- [K] J.J Kohn: Subellipticity of the $\bar{\partial}$ -Neumann problem on pseudo-convex domains: sufficient conditions. *Acta Math.* **142** (1979), no. 1–2, 79–122.
- [M] B. Malgrange: Ideals of differentiable functions. *Tata Institute of Fundamental Research Studies in Mathematics*, **3** Tata Institute of Fundamental Research, Bombay; Oxford University Press, London: 1967.
- [N] J.K. Nowak: On the real algebra of quasianalytic function germs. IMUJ Preprint 2010/08. <http://www2.im.uj.edu.pl/badania/preprinty/imuj2010/pr1008.pdf>
- [Rz] J.M. Ruiz: On Hilbert’s 17th problem and real Nullstellensatz for global analytic functions. *Math. Z.* **190** (1985), no. 3, 447–454.
- [S] Y.T. Siu: Hilbert Nullstellensatz in global complex analytic case. *Proc. Amer. Math. Soc.* **19** (1969), 296–298.
- [To] A. Tognoli: Proprietà globali degli spazi analitici reali. *Ann. Mat. Pura Appl.* **75** (1967) no. 4, 143–218.
- [WB] H. Whitney, F. Bruhat: Quelques propriétés fondamentales des ensembles analytiques réels. *Comment. Math. Helv.* **33** (1959), 132–160.

DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DEGLI STUDI DI PISA, LARGO BRUNO PONTECORVO, 5,
56127 PISA, ITALY

E-mail address: `acquistf@dm.unipi.it`, `broglia@dm.unipi.it`

DEPARTAMENTO DE ÁLGEBRA, FACULTAD DE CIENCIAS MATEMÁTICAS, UNIVERSIDAD COMPLUTENSE
DE MADRID, 28040 MADRID (SPAIN)

E-mail address: `josefer@mat.ucm.es`